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Generalizations of Integrable Localized Induction Equation for Stretched Vortex Filament

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Contents

The localized induction equation(LIE) for the vortex filament is

$$\boldsymbol{S}_t = \boldsymbol{S}_s \times \boldsymbol{S}_{ss}.$$

Here $\boldsymbol{S}(s,t)$ is a position vector (X,Y,Z) and suffices of s and t mean the partial differentiation with respect to the arclength along the filament and the time, respectively.

- LIE is derived by the Biot-Savart law with the localized induction approximation and describes the swirl flow of vortex filament.

- LIE is integrable and has N soliton solution,

- LIE ia a member of the hierarchy,

- LIE is connected to the nonlinear Schrödinger equation,

A. Sym and Jan Cieśliński, and W.K. Schief and C. Rogers study LIE from the geometrical point of view.

Definition of the local stretch

The local length dl of the vortex filament at $\boldsymbol{X} = (X, Y, Z)$ parametrized by s is given by

$$\mathrm{d}l = \sqrt{(\mathrm{d}X)^2 + (\mathrm{d}Y)^2 + (\mathrm{d}Z)^2}.$$

Express dl with s as

$$\mathrm{d}l(s) = \sqrt{(X_s)^2 + (Y_s)^2 + (Z_s)^2}\,\mathrm{d}s.$$

Then the local stretch l_s is defined by

$$l_s = \frac{\mathrm{d}l}{\mathrm{d}s} = \sqrt{(X_s)^2 + (Y_s)^2 + (Z_s)^2}.$$

$l_s = 1$ means a filament without stretch, $l_s > 1$ with stretch and $l_s < 1$ with shrink.

We consider the localized induction equation (LIE with stretch)

$$\boldsymbol{R}_t = \frac{\boldsymbol{R}_r \times \boldsymbol{R}_{rr}}{|\boldsymbol{R}_r|^3}.$$

$\boldsymbol{R}(r,t)$ is a position vector and r is a parameter along the filament. If $|\boldsymbol{R}_r| = 1$, that is, no stretch, then LIE with stretch reduces to the standard LIE.

Note that LIE with stretch is an integrable equation as well as LIE.

The inverse scattering method for LIE with stretch is given by

$$\psi_r = U\psi,$$
$$\psi_t = W\psi.$$

With a spectral parameter λ, U and W are given by

$$U = -\frac{\mathrm{i}\lambda}{2}\begin{pmatrix} Z_r & X_r - \mathrm{i}Y_r \\ X_r + \mathrm{i}Y_r & -Z_r \end{pmatrix},$$
$$W = \lambda W_{12} + \lambda^2 W_{11},$$

where $\boldsymbol{R} = (X, Y, Z)$ is the position vector. The compatibility condition is given by

$$U_t - W_r + [U, W] = 0.$$

If we take W_{11} and W_{12} as

$$W_{11} = -\frac{\mathsf{i}}{2\sqrt{(X_r^2 + Y_r^2 + Z_r^2)}} \begin{pmatrix} Z_r & X_r - \mathsf{i}Y_r \\ X_r + \mathsf{i}Y_r & -Z_r \end{pmatrix},$$

$$W_{12} = \frac{\mathsf{i}}{2(X_r^2 + Y_r^2 + Z_r^2)^{2/3}}$$

$$\begin{pmatrix} X_rY_{rr} - Y_rX_{rr} & Y_rZ_{rr} - Z_rY_{rr} - \mathsf{i}(Z_rX_{rr} - X_rZ_{rr}) \\ Y_rZ_{rr} - Z_rY_{rr} + \mathsf{i}(Z_rX_{rr} - X_rZ_{rr}) & -X_rY_{rr} + Y_rX_{rr} \end{pmatrix},$$

we obtain LIE with stretch. Then we see that LIE with stretch is integrable. In fact we do not use the condition $X_r^2 + Y_r^2 + Z_r^2 = 1$ so that LIE with stretch includes solutions for stretched and/or shrunk vortex filaments as well as unstretched ones.

Let us consider relationship between LIE and LIE with stretch where LIE is expressed as

$$\boldsymbol{S}_t = \boldsymbol{S}_s \times \boldsymbol{S}_{ss},$$

and LIE with stretch as

$$\boldsymbol{R}_t = \frac{\boldsymbol{R}_r \times \boldsymbol{R}_{rr}}{|\boldsymbol{R}_r|^3}.$$

We can proof that $|\boldsymbol{S}_s|$ and $|\boldsymbol{R}_r|$ are independent of t by taking the inner product such as

$$\frac{\partial \boldsymbol{S}}{\partial s} \cdot \frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{S}}{\partial t}\right) = 0,$$
$$\frac{\partial \boldsymbol{R}}{\partial r} \cdot \frac{\partial}{\partial r}\left(\frac{\partial \boldsymbol{R}}{\partial t}\right) = 0,$$

with the equations of motion LIE and LIE with stretch. Then $|\boldsymbol{S}_s|$ and $|\boldsymbol{R}_r|$ are a function of s and r, respectively.

Let us consider the transformation between two equations as

$$\mathrm{d}s = g\,\mathrm{d}r,$$

where g is a metric defined by

$$g = \sqrt{\frac{\partial \boldsymbol{R}}{\partial r} \cdot \frac{\partial \boldsymbol{R}}{\partial r}}.$$

g is a function of r and also represents the local stretch. With the metric we see that LIE is transformed into LIE with stretch.

Using the metric we have

$$s = f(r).$$

We can obtain a solution of LIE with stretch by substituting $f(r)$ into s of LIE such as

$$\boldsymbol{R}(r,t) = \boldsymbol{S}(f(r),t).$$

With N soliton solution of $\boldsymbol{S}$, we can obtain N soliton solution of $\boldsymbol{R}$.

Introducing the inverse function $r = h(s)$ such as

$$f(h(s)) = s,$$

we see that LIE with stretch becomes LIE.

One vortex soliton solution of LIE with stretch is given by

$$R_x = \frac{\lambda_I}{\lambda_R^2 + \lambda_I^2} \sin 2(\lambda_R f(r) - \omega_R t) \mathrm{sech} 2(\lambda_I f(r) - \omega_I t),$$

$$R_y = -\frac{\lambda_I}{\lambda_R^2 + \lambda_I^2} \cos 2(\lambda_R f(r) - \omega_R t) \mathrm{sech} 2(\lambda_I f(r) - \omega_I t),$$

$$R_x = f(r) - \frac{\lambda_I}{\lambda_R^2 + \lambda_I^2} \tanh 2(\lambda_I f(r) - \omega_I t),$$

where $\lambda = \lambda_R + \mathrm{i}\lambda_I$ and $\omega = 2\lambda^2$.

Stretched vortex soliton with a factor A as

$$S_x = A\frac{\lambda_I}{\lambda_R^2+\lambda_I^2}\sin(2\Omega)\operatorname{sech}(2\Theta),$$
$$S_y = -A\frac{\lambda_I}{\lambda_R^2+\lambda_I^2}\cos(2\Omega)\operatorname{sech}(2\Theta),$$
$$S_z = s-\frac{\lambda_I}{\lambda_R^2+\lambda_I^2}\tanh(2\Theta),$$

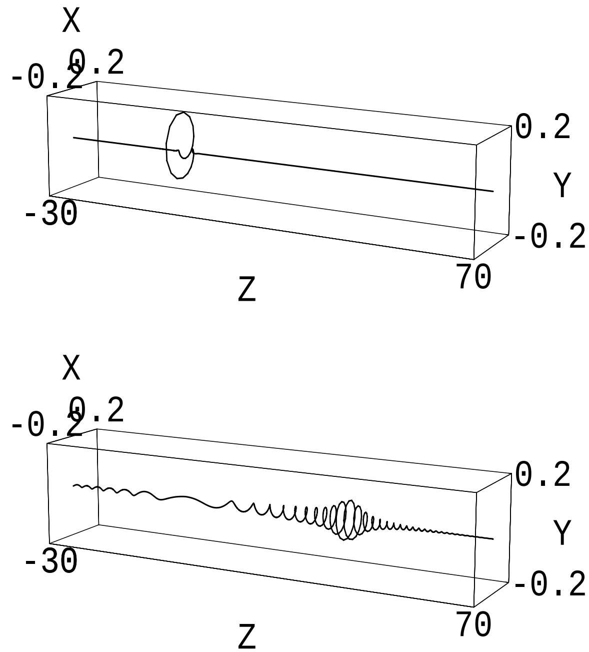
where

$$\Omega = \lambda_R s - 2(\lambda_R^2-\lambda_I^2)t, \qquad \Theta = \lambda_I s - 4\lambda_R\lambda_I t.$$

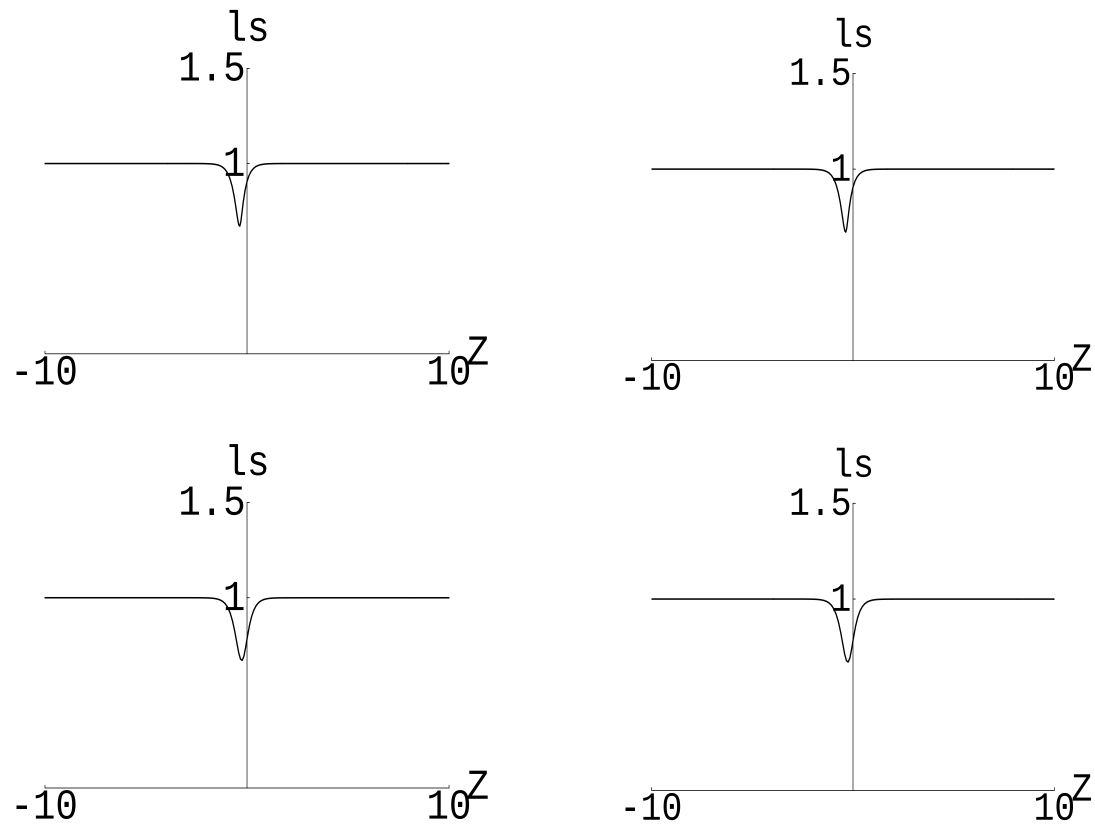
The local stretch is given by

$$l_s^2 = 1+\frac{4\lambda_I^2}{\lambda_R^2+\lambda_I^2}(A^2-1)[\operatorname{sech}^2(2\Theta)-\frac{\lambda_I^2}{\lambda_R^2+\lambda_I^2}\operatorname{sech}^4(2\Theta)].$$

If $A=1$, there is no stretch and if $A\neq 1$, then local stretch for $A>1$ and local shrink for $A<1$.



Time evolution of shrunk vortex soliton with LIE (left) and LIE with stretch (right) at $t = 0$, 6 for $A = 0.6$ and $\lambda = 1.5 + \mathrm{i}$.



Local stretch of shrunk vortex soliton with LIE (left) and LIE with stretch (right) at $t = 0$, 6 for $A = 0.6$ and $\lambda = 1.5 + \mathrm{i}$.

Loop type of stretched vortex soliton

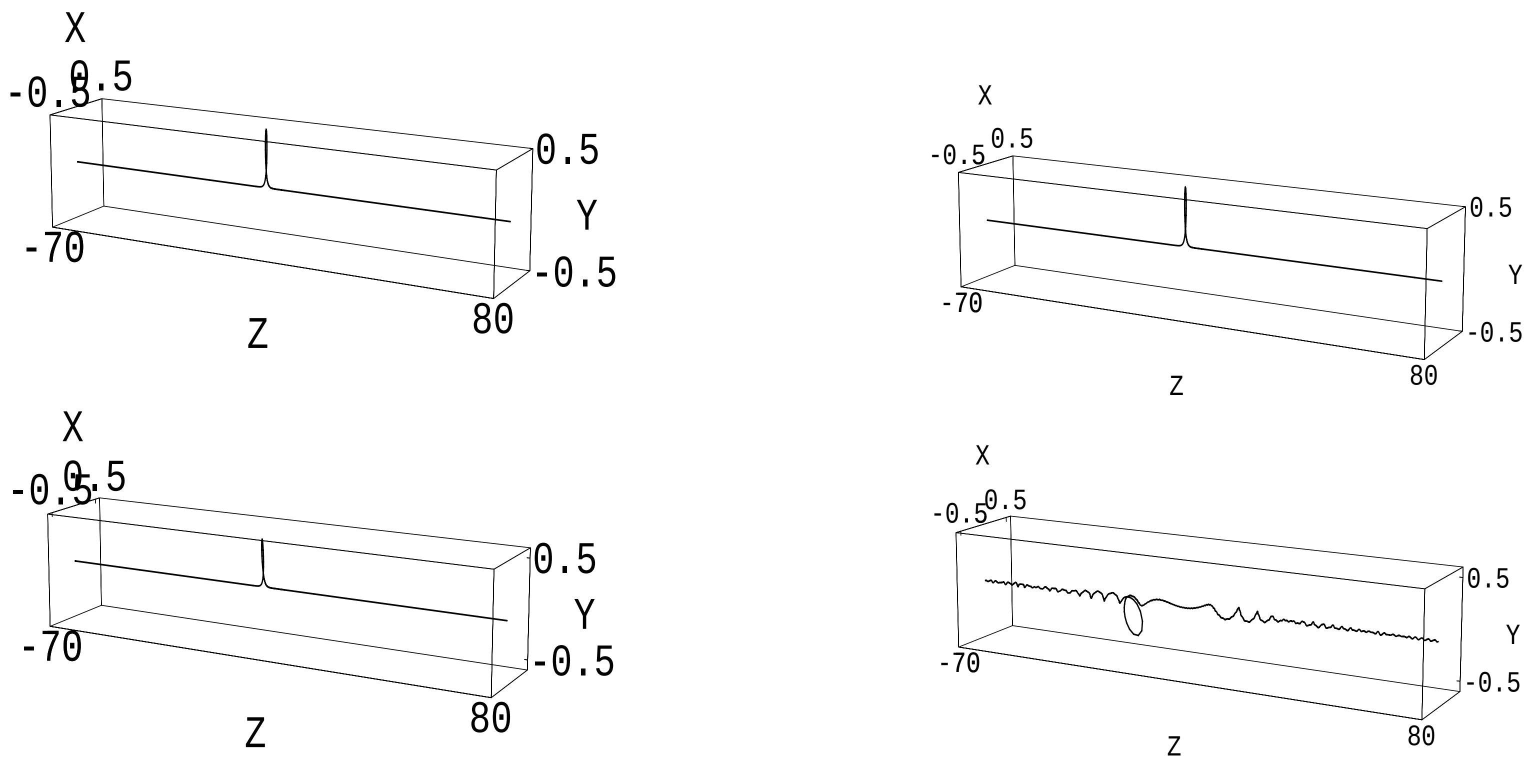
$$S_x = -A \sin 4t \,\mathrm{sech} 2s,$$

$$S_y = A \cos 4t \,\mathrm{sech} 2s,$$

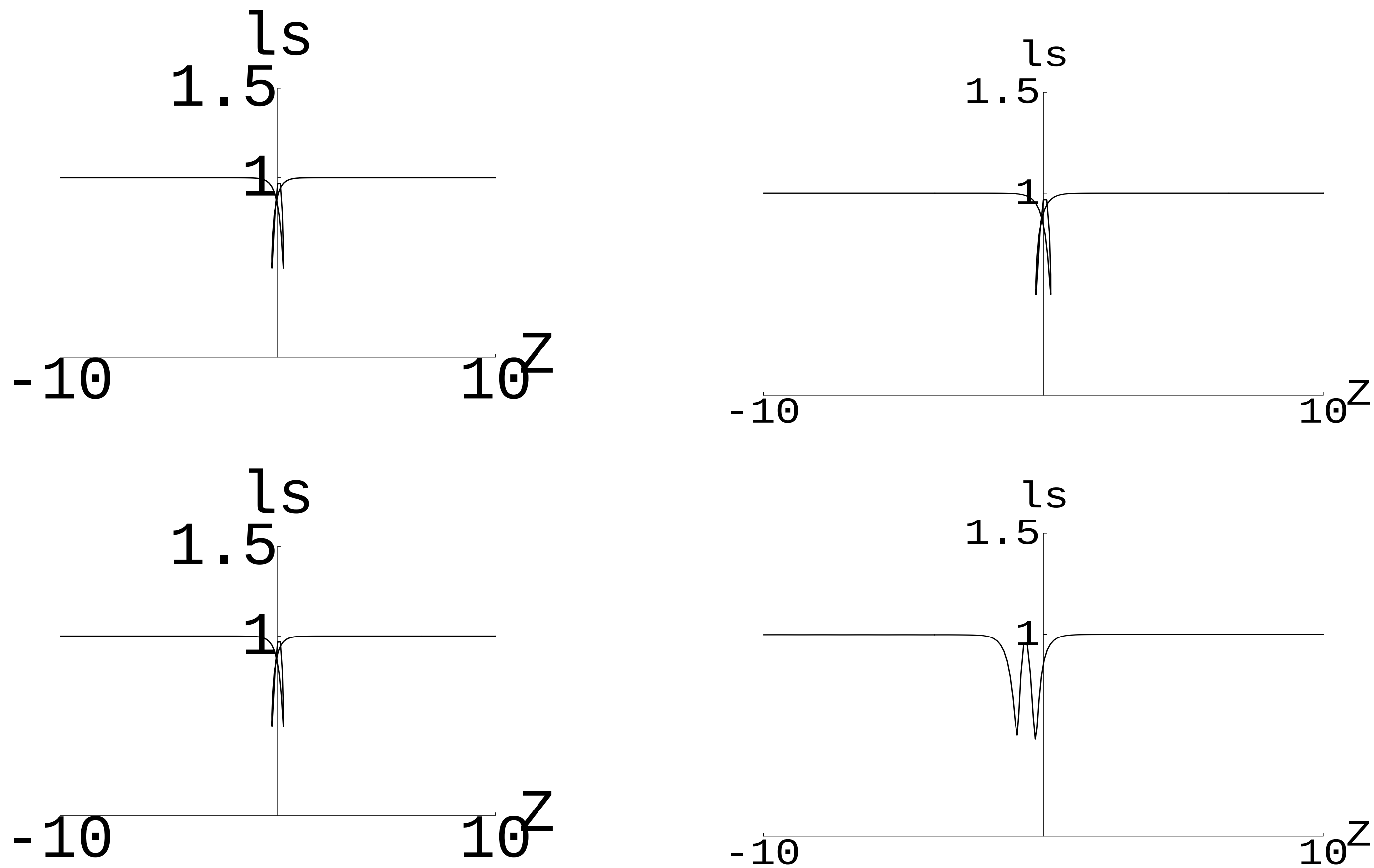
$$S_z = s - \tanh 2s,$$

which is exact solution of LIE, but is not solution of LIE with stretch. Here A is a factor to represent stretch of the soliton. The local stretch is given by

$$l_s^2 = 1 + 4(A^2 - 1)\mathrm{sech}^2 2s \tanh^2 2s.$$



Time evolution of shrunk loop soliton with LIE (left) and LIE with stretch (right) at $t = 0, 6$ for $A = 0.5$.



Local stretch of shrunk loop soliton with LIE (left) and LIE with stretch (right) at $t = 0$, 6 for $A = 0.5$.

We can generalize the metric such as

$$\mathrm{d}s = g^n \, \mathrm{d}r,$$

then, we can obtain a generalized localized induction equation

$$\boldsymbol{R}_t = \frac{\boldsymbol{R}_r \times \boldsymbol{R}_{rr}}{|\boldsymbol{R}_r|^{3n}},$$

which is still an integrable equation.

According to the above results, we find a further generalization of LIE by introducing the independent variable transformation as

$$s = f(r)$$

such that

$$\mathrm{d}s = \frac{\mathrm{d}f(r)}{\mathrm{d}r}\mathrm{d}r \equiv g\mathrm{d}r.$$

If we assume that

$$\boldsymbol{R}(r,t) = \boldsymbol{S}(f(r),t),$$

then LIE reduces to

$$\boldsymbol{R}_t = \frac{\boldsymbol{R}_r \times \boldsymbol{R}_{rr}}{g^3}.$$

Summary

1. We have shown the relationship between LIE and LIE with stretch by using the metric $g(r)$ and the inverse transformation $r = h(s)$.

2. We have obtained N vortex soliton solution of LIE with stretch by using N soliton solution of LIE and shown explicitly one soliton solution.

3. We have shown some numerical results by using LIE without and with stretch.

4. Further generalizations of LIE have been found where the integrability of the reduced equation is preserved.

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Localized-Induction Concept on a Curved Vortex and Motion of an Elliptic Vortex Ring

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The localized-induction concept for the induction effect of a smooth curved vortex on itself is derived. This is an approximation applicable to the limiting case of a vortex filament of infinitesimal core size and of negligible long-distance effect, and was already successfully utilized in the investigations of the motion and deformation of a curved vortex filament given various initial configurations. Two theorems obtained under this concept are that the arc length of a vortex filament and the projected area of a closed vortex filament are both invariant with respect to time. These theoretical predictions are examined by a numerical analysis of the motion of an initially plane elliptic vortex ring of various eccentricities.

II. LOCALIZED-INDUCTION CONCEPT

We are concerned with the induced velocity on the vortex by itself. Then

$$\mathbf{r}_{ij}(\xi, t) = \mathbf{r}_i(s_i, t) - \mathbf{r}_j(s_i + \xi, t), \quad (3)$$

where ξ is the running parameter along the vortex. ξ being small, $\mathbf{r}_{ij}$ may be expanded in a Taylor series:

$$\mathbf{r}_{ij}(\xi) = \mathbf{a}_1\xi + \mathbf{a}_2\xi^2 + \cdots \quad (4)$$

in which

$$\mathbf{a}_1 = \partial\mathbf{r}_{ij}/\partial\xi, \quad \mathbf{a}_2 = \tfrac{1}{2}\,\partial^2\mathbf{r}_{ij}/\partial\xi^2, \ \ldots \text{ at } \xi = 0. \quad (5)$$

Here the smoothness of the curve is assumed so that the derivatives of the curve exist. Then

$$\partial\mathbf{r}_{ij}/\partial s_i = \partial\mathbf{r}_{ij}/\partial\xi = \mathbf{a}_1 + 2\mathbf{a}_2\xi + \cdots$$

and

$$\begin{aligned} -\partial\mathbf{r}_{ij}/\partial s_i \times \mathbf{r}_{ij} &= (\mathbf{a}_1\xi + \mathbf{a}_2\xi^2 + \cdots) \times (\mathbf{a}_1 + 2\mathbf{a}_2\xi + \cdots) \\ &= (\mathbf{a}_1 \times \mathbf{a}_1)\xi + (\mathbf{a}_2 \times \mathbf{a}_1 + 2\mathbf{a}_1 \times \mathbf{a}_2)\xi^2 + O(\xi^3) \\ &= (\mathbf{a}_1 \times \mathbf{a}_2)\xi^2 + O(\xi^3) \\ &= (\mathbf{a}_1 \times \mathbf{a}_2)\,|\xi|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\mathbf{r}_{ij}|^2 &= |(\mathbf{a}_1\xi + \mathbf{a}_2\xi^2 + \cdots)^2| \\ &= |\mathbf{a}_1|^2\,|\xi|^2 + 2\mathbf{a}_1\cdot\mathbf{a}_2\xi^3 + \cdots, \\ r_{ij} &= |\mathbf{a}_1|\,|\xi|\left(1 + \frac{\mathbf{a}_1\cdot\mathbf{a}_2}{|\mathbf{a}_1|^2}\,\xi + \cdots\right), \end{aligned}$$

and

$$r_{ij}^{-3} = |\mathbf{a}_1|^{-3}\,|\xi|^{-3}\left(1 - 3\,\frac{\mathbf{a}_1\cdot\mathbf{a}_2}{|\mathbf{a}_1|^2}\,\xi + \cdots\right).$$

Therefore

$$\mathbf{q}_i = \frac{\kappa}{4\pi}\int\left[\frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1|^3}\,\frac{1}{|\xi|} + O(1)\right]d\xi.$$

When the integration is made over the limits $\epsilon \le |\xi| < 1$, one obtains

$$\mathbf{q}_i = \frac{\kappa}{2\pi}\,\frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1|^3}\log\left(\frac{1}{\epsilon}\right) + O(1),$$

or

$$\frac{4\pi\mathbf{q}_i}{\kappa} = \frac{(\partial\mathbf{r}/\partial s)_i \times (\partial^2\mathbf{r}/\partial s^2)_i}{|(\partial\mathbf{r}/\partial s)_i|^3}\log\left(\frac{1}{\epsilon}\right) + O(1).$$

If $\epsilon \ll 1$, i.e., the limiting case of a vortex filam[...] the term which is order unity may be neglect[...] which corresponds to an omission of long-dista[...] effects.

Under this approximation, which might [...] propriately be called the localized-induction conc[...] the velocity induced on a curved vortex by [...] may be written, after pertinent definitions of [...] and length, as

$$\frac{\partial\mathbf{r}}{\partial t} = \frac{(\partial\mathbf{r}/\partial s) \times (\partial^2\mathbf{r}/\partial s^2)}{|(\partial\mathbf{r}/\partial s)|^3}.$$

It is noted that the induced velocity on a cu[...] vortex is indeed proportional to the local curva[...] of the vortex and therefore that Hama's intu[...] theorem[3] concerning the maxima of the self-indu[...] velocity is proved exact in this approxima[...]

Paper by Arms and Hama

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