A fiber structure of Teichmüller space and conformal field theory

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Introduction

Conformal Field Theory (CFT):

- Special class of 2D quantum field theories.
- Mathematical definition (G. Segal, Kontevich \( \approx 1986 \))
- Deeply connected to algebra, topology and analysis.
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Complex analysis/geometry:
- \((\infty\text{-dim})\) moduli space of Riemann surfaces
- Sewing=gluing=welding
- Quasiconformal mappings
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Our General Aim:
- Provide a natural analytic setting for the rigorous definition of CFT in higher genus. Definitions and Theorems.
- Use CFT ideas (especially sewing) to prove new results in Teichmüller theory and geometric function theory.
Motivation/Application: Conformal Field Theory

\[ H \otimes H \xrightarrow{A([\Sigma, \psi])} H \otimes H \otimes H \]

\[ \psi_1^\circ \]

\[ \psi_2^\circ \]

\[ \psi_3^\circ \]

\[ S^1 \]

\[ S^1 \]

\[ S^1 \]

\[ S^1 \]

\[ \Sigma \]

\[ \psi_{\text{in}}^1 \]

\[ \psi_{\text{in}}^2 \]

\[ \psi_{\text{in}}^3 \]

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Quasiconformal Maps I

\( f : \Omega \subset \mathbb{C} \rightarrow \mathbb{C}. \) Homeomorphism. Orientation Preserving.

\[ \text{Jacobian}(f) = \cdots = |f_z|^2 - |f_{\bar{z}}|^2 > 0. \] So, \( |f_{\bar{z}}/f_z| < 1. \)
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Complex Dilatation = \mu(z) = f_{\bar{z}}/f_z.

Circular Dilatation = \frac{\text{major axis}}{\text{minor axis}} = \frac{1+|\mu|}{1-|\mu|}.

Note: $f(z)$ conformal $\iff f_{\bar{z}} = 0 \iff \mu(z) = 0 \iff \text{Circ.Dil.} = 1.$
Quasiconformal Maps II

\[ f : \Omega \subset \mathbb{C} \rightarrow \mathbb{C}, \quad \mu(z) = \frac{f \bar{z}}{f z}. \quad \text{Circular Dilatation} = \frac{1 + |\mu|}{1 - |\mu|}. \]
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Geometric Definition:

\( f \) is \textit{\( K \)-quasiconformal} if its circular dilatation is globally bounded by \( K \). (i.e. Infinitesimally, circles map to ellipses of bounded eccentricity).
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Geometric Definition:

\( f \) is **K-quasiconformal** if its circular dilatation is globally bounded by \( K \). (i.e. Infinitesimally, circles map to ellipses of bounded eccentricity).

Analytic Definition:

\( f \) is **K-quasiconformal** if it satisfies the Beltrami Equation

\[
\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}
\]

for some \( \mu(z) \) with \( \|\mu\|_{\infty} = k < 1. \quad K = \frac{1 + k}{1 - k} \).

**Note:** Technical conditions skipped. QC maps are only differentiable almost everywhere etc.
Basic Objects

- Riemann Surfaces with boundary
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- Fix: \( g = \text{genus}, \ n = \# \text{of boundary components} \). The **moduli space** is the space of conformal equivalence classes of surfaces.

- Quasiconformal map
- Quasisymmetric map

**Definition:**

\[
h : S^1 \rightarrow S^1
\]

\( h \) has quasiconformal extensions to \( \mathbb{C} \).
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- Quasiconformal map
- Quasisymmetric map
- Quasisymmetric boundary parametrization
Definition (Teichmüller space):
\[ T(\Sigma) = \{ (\Sigma_1, f, \Sigma) \} / \sim. \]
\[ (\Sigma_1, f, \Sigma) \sim (\Sigma_2, g, \Sigma) \iff \exists \text{ conformal } \sigma : \Sigma_1 \to \Sigma_2 \text{ such that } g^{-1} \circ \sigma \circ f \approx \text{id (rel. boundary)} \]

Teichmüller metric:
\[ \text{distance} \left[ (\Sigma_1, f, \Sigma), (\Sigma_2, g, \Sigma) \right] = \inf_{f, g} \log \left( \text{circular dilatation of } g \circ f - 1 \right) \]

This measures how close (in the quasiconformal sense) to a conformal map there is from \( \Sigma_1 \) to \( \Sigma_2 \).

Fix a base Riemann surface \( \Sigma \).
Given \( \Sigma_1 \) and quasiconformal \( f : \Sigma \to \Sigma_1 \), write \( (\Sigma, f, \Sigma_1) \).
Teichmüller Theory

Definition and Facts

Teichmüller Space = space of Riemann surfaces

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Teichmüller metric:

\[ \text{distance}([\Sigma, f, \Sigma_1], [\Sigma, g, \Sigma_2]) = \inf_{f,g} \log(\text{circular dilatation of } g \circ f^{-1}) \]

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Teichmüller space facts

Fix $\Sigma$. \( f : \Sigma \to \Sigma_1 \). \( T(\Sigma) = \) Teichmüller space.

Why?

- $\mu(f) = f_\bar{z}/f_z$ is a differential form on the base surface.
- Study the Teichmüller space by studying certain spaces of forms.
- This is classical work from the 50’s and 60’s of Ahlfors and Bers et al. Well developed theory.
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Classical Facts:

1. $T($torus$) =$ upper half-plane.
2. If $\Sigma$ is closed (with punctures) then $T^P(\Sigma)$ is a finite-dimensional complex manifold.
3. If $\Sigma$ is a surface with boundary then $T^B(\Sigma)$ is an $\infty$-dimensional complex manifold.
4. Moduli space $= T(\Sigma)/$ (Mapping Class Group).
5. The moduli space is not a manifold.
Sewing

\[ \Sigma_1 \# \Sigma_2 = (\Sigma_1 \sqcup \Sigma_2) / (\psi_1(x) = \psi_2(y)) \]

Note: If \( \psi_i \) are conformal then \( \Sigma_1 \# \Sigma_2 \) immediately becomes a Riemann surface. This is what was previously used in CFT.

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Sewing

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Conformal Welding

\[ \Delta - \text{unit disk, } \Delta^* = \hat{\mathcal{C}} \setminus \bar{\Delta}, \quad h : S^1 \to S^1 \text{ (quasisymmetry)} \]

Theorem (conformal welding:)

There exists conformal maps \( F_1 \) and \( F_2 \) such that \( F_2^{-1} \circ F_1 = h \) on \( S^1 \).
Quasisymmetric Sewing

\( \psi_1 \) and \( \psi_2 \) – quasisymmetric boundary parametrizations. Define charts on \( \Sigma_1 \# \Sigma_2 \) by:
Quasisymmetric Sewing

$\psi_1$ and $\psi_2$ – quasisymmetric boundary parametrizations.
Define charts on $\Sigma_1 \# \Sigma_2$ by:

Proposition (R-S 06)
This gives the unique complex structure on $\Sigma_1 \# \Sigma_2$ which is compatible with $\Sigma_1$ and $\Sigma_2$. 
Holomorphicity of sewing

Key idea:
Fix $\tau$ to be a quasisymmetric boundary parametrization of $\Sigma$. 
$[\Sigma, f, \Sigma_1] \in T^B(\Sigma)$ contains boundary parametrization information for $\Sigma_1$ via $\psi = \tau \circ f^{-1}$.

Theorem (R-S 2006)
The sewing operations are holomorphic. That is,

$$T^B(\Sigma_1) \times T^B(\Sigma_2) \xrightarrow{\text{sew}} T^B(\Sigma_1 \# \Sigma_2)$$

is holomorphic.
Cap Sewing: $T^B \to T^P$

**Theorem (RS 08)**

1. $T^B$ is a holomorphic fiber space over $T^P$.
2. The fibers are complex Banach manifolds modeled on $\mathcal{O}_{qc} = \{ f : \mathbb{D} \to \mathbb{C} \mid f \text{ is univalent, has qc extension, and } f(0) = 0. \}$
HELP!!

- In one (and several) complex variables an injective holomorphic map automatically has a holomorphic inverse.
- This is not true in infinite dimensions in general.
- In the Banach space setting do there exist nice conditions to guarantee holomorphicity of the inverse?