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Outline

1: Review: KP and 2-Toda $\tau$ functions.
- KP $\tau$ functions
- Hilbert space Grassmannian and linear group actions
- The $\tau$ function as a determinant
- Examples of $\tau$ functions
  - Schur functions
  - Orthogonal polynomials
- Fermionic Fock space
- Schur function expansions

2: Convolution symmetries
- Representation on $\mathcal{H}$
- Fock space representation of convolution symmetries
- Effect of convolution symmetries on $\tau$-functions
- Applications to matrix models
  - New matrix models as $\tau$ functions
- Convolution flows and the $Q$ operator
- Triangular boundary operators $\hat{Q}(q), \hat{Q}(\tilde{q})$ of Toeplitz type
A KP tau function $\tau(t)$ is a function of an infinite set of flow variables $t = (t_1, t_2, \ldots)$, satisfying an infinite set of bilinear equations, the Hirota Bilinear equations:

$$\text{res}_{z=0} \left( \psi^+(z, t)\psi^-(z, t + s) \right) = 0,$$

(identically in $s := (s_1, s_2, \ldots)$), where the Baker-Akhiezer function $\psi^+(z, t)$ and its dual $\psi^-(z, t)$ are defined by the Sato formula:

$$\psi^\pm(z, t) := e^{\pm \sum_{i=1}^\infty t_iz^i} \times \frac{\tau(t \mp [z^{-1}])}{\tau(t)}$$

$$[z^{-1}] := \left( \frac{1}{z}, \frac{1}{2z^2}, \ldots \right)$$

**Question**: How to construct such $\tau$ functions? What do they mean?
Hilbert Space Grassmannians

Model for Hilbert space

\[ \mathcal{H} := L^2(S^1) = \mathcal{H}_+ + \mathcal{H}_-, \]
\[ \mathcal{H}_+ = \text{span}\{z^i\}_{i \in \mathbb{N}}, \quad \mathcal{H}_- = \text{span}\{z^{-i}\}_{i \in \mathbb{N}^+}, \]

The Sato-Segal-Wilson Grassmannian is defined as

\[ \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) = \{ \text{closed subspaces } w \subset \mathcal{H} \text{ “commensurable” with } \mathcal{H}_+ \} \]
i.e., such that orthogonal projection to \( \mathcal{H}_+ \) along \( \mathcal{H}_- \)

\[ \pi_{\perp} : w \rightarrow \mathcal{H}_+ \]
is a Fredholm map and orthogonal projection to \( \mathcal{H}_- \)

\[ \pi_{\perp} : w \rightarrow \mathcal{H}_- \]
is “small” (e.g., Hilbert-Schmidt). \((\mathcal{H}_+ \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \text{ is the “origin”}).\)
Basis labelling and frames

Orthonormal basis for $\mathcal{H}$:

$$\{ e_i := z^{-i-1} \}_{i \in \mathbb{Z}},$$

In terms of frames, let

$$w = \text{span}\{ w_1, w_2, \ldots \},$$

and expand the basis vectors $w_i$ in the orthonormal basis $\{ e_j \}$

$$w_i := \sum_{j \in \mathbb{Z}} W_{ji} e_j.$$

Define doubly $\infty$ column vectors $\{ W_i \}_{i=1,2\ldots}$ with components

$$(W_i)_j := W_{ji}$$

and the rectangular $2\infty \times \infty$ matrix $W$ with columns $\{ W_i \}_{i=1,2\ldots}$

$$W := (W_1, W_2, \cdots)$$
Abelian group actions: \( \Gamma_\pm \times \mathcal{H} \rightarrow \mathcal{H} \):

\( \Gamma_\pm := \{ \gamma_\pm(t) := e^{\sum_{i=1}^{\infty} tiz^{\pm i}} \} \)

\((\gamma_\pm(t), f \in L^2(S^1)) \mapsto \gamma_\pm(t)f\)

This induces an action on frames \( W \), for \( w \in Gr_{\mathcal{H}_+}(\mathcal{H}) \)

\( \gamma_\pm(t) \times W \mapsto W(t) := e^{\sum_{i=1}^{\infty} t_i \Lambda^{\pm i}} W \)

where

\( \Lambda(e_i) = e_{i-1} \)

More generally, we have the **general linear group action**:

\( GL(\mathcal{H}) \times Gr_{\mathcal{H}_+}(\mathcal{H}) \rightarrow Gr_{\mathcal{H}_+}(\mathcal{H}) \)

\((g \in GL(\mathcal{H}), W) \mapsto gW\)

represented by doubly infinite, invertible matrices

\( g = e^A, \quad A \in gl(\infty). \quad A = (A_{ij})|_{i,j, \in \mathbb{Z}} \)
For $w \in Gr_{\mathcal{H}^+}(\mathcal{H})$, the KP $\tau$ function $\tau_w(t)$ is obtained as the Fredholm determinant of the orthogonal projection of $W(t)$ to $\mathcal{H}^+$:

$$\tau_w(t) = \det(\pi^\perp : W(t) \to \mathcal{H}^+), \quad t = (t_1, t_2, \ldots)$$

or, equivalently if

$$W(t) = \begin{pmatrix} W_+(t) \\ W_-(t) \end{pmatrix}$$

then

$$\tau_w(t) = \det W_+(t).$$
Example: 1. Schur functions ("elementary building blocks")

Consider **Partitions**: 

$$\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}), \quad \lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)}, \quad \lambda_i \in \mathbb{N}^+$$

of length $\ell(\lambda)$ and weight $|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i$

Define $w_\lambda \in Gr_{H_+}(\mathcal{H})$ as

$$w_\lambda := \text{span}\{e_{\lambda_i-i}\}$$

Then

$$\tau_{w_\lambda}(t) = s_\lambda(t)$$

where the **Schur function**

$$s_\lambda(t) := \text{tr}(\rho_\lambda(g)), \quad g \in GL(N)$$

$$t := (t_1, t_2, \cdots), \quad t_i := \frac{1}{i} \text{tr}(g^i), \quad g \in GL(N)$$

is the **character** of the irreducible representation

$$\rho_\lambda : GL(N) \longrightarrow \text{End}(T^{(\lambda)} \subset (\mathbb{C}^N)^{\otimes |\lambda|})$$

obtained by restricting to tensors of symmetry type $\lambda$. 
Example: 2. Orthogonal polynomials and Random Matrix integrals

Let

\[ w_{d\mu} = \text{span}\left\{ \frac{1}{z^N} p_{N+i} \right\}_{i=0,1,2,...} \in \text{Gr}_{\mathcal{H}+}(\mathcal{H}) \]

where \( \{p_i(z)\}_{i \in \mathbb{N}} \) are orthogonal polynomials with respect to a measure \( d\mu(z) \) on some set of curve segments \( \Gamma \) in the complex plane (e.g., the real line \( \mathbb{R} \))

\[ \int p_i(z)p_j(z)d\mu(z) = \delta_{ij} \]

Then

\[ \tau_{w_{d\mu}}(t) = \prod_{a=1}^{N} \int_{\Gamma} d\mu(z_a)e^{\sum_{i=1}^{\infty} t_i z_i^a} \Delta^2(z) \]

where \( \Delta(z) = \prod_{a<b}^{N}(z_a - z_b) \) (Vandermonde determinant)
Random matrix integrals

By the **Weyl integral formula** on $U(N)$, we have

$$
\tau_{Wd\mu}(t) \propto Z_{N,f}(t) := \int_{H_{N\times N}} d\mu_{N,f}(M, t)
$$

where

$$
d\mu_{N}(M, t) := d\mu_{N}(M) e^{\text{tr}(\sum_{i=1}^{\infty} t_i M_i)}
$$

is a deformation family of $U(N)$ conjugation invariant measures on the space $H_{N\times N}$ of Hermitian $N \times N$ matrices.

$$
d\mu_{N}(UMU^\dagger) = d\mu_{N}(M), \quad \forall U \in U(N), \quad M \in H_{N\times N}
$$
**Fermionic Fock space** $\mathcal{F}$

For every partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ and integer $N \in \mathbb{Z}$ define the extended semi-infinite sequence

$$\lambda = (\lambda_1, \ldots \lambda_{\ell(\lambda)}, 0, 0, \ldots)$$

and "particle positions"

$$l_j := \lambda_j - j + N$$

The **fermionic Fock space** $\mathcal{F}$ is the **exterior space** (orthogonal direct sum of charge $N$ subspaces)

$$\mathcal{F} := \wedge \mathcal{H} = \bigoplus_{N \in \mathbb{Z}} \mathcal{F}_N.$$  

spanned by semi-infinite wedge products (orthonormal basis for $\mathcal{F}_N$)

$$|\lambda, N\rangle := e_{l_1} \wedge e_{l_2} \wedge \cdots$$

Each charge $N$ sector $\mathcal{F}_N$ has a charged **vacuum vector**

$$|0, N\rangle = e_{N-1} \wedge e_{N-2} \wedge \cdots,$$
Fermionic creation and annihilation operators

In terms of the **Orthonormal basis for** $\mathcal{H}$, and **dual basis for** $\mathcal{H}^*$

$$\{ e_i := z^{-i-1} \}_{i \in \mathbb{Z}}, \quad \{ \tilde{e}_i \}_{i \in \mathbb{Z}}, \quad \tilde{e}_i(e_j) = \delta_{ij}$$

define the Fermi **creation and annihilation operators** (exterior and interior multiplication):

$$\psi_i v := e_i \wedge v, \quad \psi_i^\dagger v := i\tilde{e}_i v, \quad v \in \mathcal{H}.$$ 

These satisfy the usual anti-commutation relations

$$[\psi_i, \psi_j]_+ = [\psi_i^\dagger, \psi_j^\dagger]_+ = 0, \quad [\psi_i, \psi_j^\dagger]_+ = \delta_{ij}.$$ 

determining the $\infty$ dimensional Clifford algebra of fermionic operators.
Plücker map and Plücker coordinates

The **Plücker map** $\mathcal{P} : \text{Gr}_{\mathcal{H}^+}(\mathcal{H}) \to \mathbb{P}(\mathcal{F})$ into the projectivization of $\mathcal{F}$,

$$\mathcal{P} : \text{span}(w_1, w_2, \ldots) \mapsto [w_1 \wedge w_2 \wedge \cdots],$$

embeds $\text{Gr}_{\mathcal{H}^+}(\mathcal{H})$ in $\mathbb{P}(\mathcal{F})$ as the intersection of an infinite number of quadrics. If orthogonal projection to $\mathcal{H}^+$

$$\pi^\perp : w \to \mathcal{H}^+$$

has Fredholm index $N$, is in the charge $N$ sector $\mathcal{P}(w) \subset \mathcal{F}_N$. Expanding in the standard orthonormal basis,

$$\mathcal{P}(w) = w_1 \wedge w_2 \wedge \cdots = \sum_{\lambda} \pi_{\lambda}(w, N) |\lambda, N >,$$

the coefficients $\pi_{\lambda}(w, N)$ are the **Plücker coordinates** of $w$ (which satisfy the infinite set of bilinear Plücker equations.)
Fermionic representation of group actions and flows

The **Plücker map**

\[ \mathcal{P} : \text{Gr}_{\mathcal{H}^+}(\mathcal{H}) \to \mathbb{P}(\mathcal{F}) \]

interlaces the action of the abelian groups

\[ \Gamma_{\pm} \times \text{Gr}_{\mathcal{H}^+}(\mathcal{H}) \to \text{Gr}_{\mathcal{H}^+}(\mathcal{H}) \]

with the following representations on \( \mathcal{F} \) (and its projectivization)

\[ \gamma_{\pm}(t) : v \mapsto \hat{\gamma}_{\pm}(t)v, \quad \hat{\gamma}_{\pm}(t) := e^{\sum_{i=1}^{\infty} t_i J_{i,\pm}}, \quad v \in \mathcal{F} \]

where

\[ J_i := \sum_{n \in \mathbb{Z}} \psi_n \psi_{n+i}^\dagger, \quad i \in \mathbb{Z} \]

More generally, if \( g = e^A \in GL(\mathcal{H}), A \in \mathfrak{gl}(\mathcal{H}) \) has the fermionic representation

\[ \hat{g} := e^{\sum_{i,j \in \mathbb{Z}} A_{ij} : \psi_i \psi_j^\dagger} : \]
Fermionic representation of KP-chain and 2-Toda $\tau$ function

For $w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) = g(\mathcal{H}_+)$, $g \in GL(\mathcal{H})$, with $\mathcal{P}(w) \subset \mathcal{F}_N$ in the charge-$N$ sector, the KP chain $\tau$-function has the fermionic representation:

$$\tau_w(t, N) = \langle N | \hat{\gamma}_+(t) \hat{g} | N \rangle =: \tau_g(t, N)$$

Similarly, for the 2-Toda $\tau$ function:

$$\tau_w^{(2)}(t, \tilde{t}, N) = \langle N | \hat{\gamma}_+(t) \hat{g} \hat{\gamma}_-(\tilde{t}) | N \rangle := \tau_g^{(2)}(t, \tilde{t}, N)$$
Review: KP and 2-Toda \( \tau \) functions. Fermionic Fock space

**Schur function expansions**

It follows that we have the **Schur function expansions**

\[
\tau_g(t, N) = \sum_\lambda \pi_\lambda(g(H_+), N)s_\lambda(t),
\]

\[
\tau_g^{(2)}(t, \tilde{t}, N) = \sum_\lambda \sum_\mu B_{\lambda, \mu}(g, N)s_\lambda(t)s_\mu(\tilde{t}).
\]

where

\[
\pi_\lambda(g(H_+), N) = \langle \lambda, N| \hat{g} | N \rangle
\]

\[
B_{\lambda, \mu}(g, N) = \langle \lambda, N| \hat{g} | \mu, N \rangle
\]

are the Plücker coordinates along the basis direction \( |\lambda, N \rangle \).
2. Convolution symmetries

Given an infinite sequence of complex numbers $T = \{ T_i \}_{i \in \mathbb{Z}}$, define

$$
\rho_i := e^{T_i}, \quad r_i := e^{T_{i-1}} - T_{i-1}, \quad i \in \mathbb{Z}.
$$

Assume the series $\sum_{i=1}^{\infty} T_{-i}$ converges and

$$
\lim_{i \to \infty} |r_i| = r \leq 1,
$$

The two series

$$
\rho_+ (z) = \sum_{i=0}^{\infty} \rho_{-i-1} z^i, \quad \rho_- (z) = \sum_{i=1}^{\infty} \rho_{i-1} z^{-i},
$$

then define analytic functions $\rho_\pm (z)$ in these regions and

$$
R_{\rho} := \prod_{i=1}^{\infty} \rho_{-i} < \infty
$$
Convolution symmetries (cont’d)

If $w \in L^2(S^1)$ has the Fourier series decomposition

$$w(z) = \sum_{i=-\infty}^{\infty} w_i z^{-i-1} = w_-(z) + w_+(z)$$

$$w_-(z) = \sum_{i=1}^{\infty} w_{i+1} z^{-i}, \quad w_+(z) = \sum_{i=0}^{\infty} w_{-i-1} z^i$$

Define the bounded linear map $C(T) : L^2(S^1) \to L^2(S^1)$

$$C(T)(w)(z) = \sum_{i=-\infty}^{\infty} \rho_i w_i z^{-i-1} = \sum_{i=-\infty}^{\infty} \rho_i w_i e_i.$$ 

so each basis element $e_i$ is multiplied by $e^{T_i}$.

The group of **Convolution Symmetries** $C(T) : \mathcal{H} \to \mathcal{H}$ is represented in the standard monomial basis $\{e_i\}$ by the diagonal matrix

$$C(T) := \text{diag}\{e^{T_i}\}.$$
Fock space representation

This abelian subalgebra of $\mathfrak{gl}(\mathcal{H})$ is generated by the operators

$$K_i := :\psi_i\psi_i^\dagger: = \begin{cases} \psi_i\psi_i^\dagger & \text{if } i \geq 0 \\ -\psi_i^\dagger\psi_i & \text{if } i < 0, \end{cases}$$

$$[K_i, K_j] = 0, \quad i, j \in \mathbb{Z}.$$ 

Define

$$\hat{C}(T) := e^{\sum_{i=-\infty}^{\infty} T_i K_i}.$$ 

Then $\hat{C}(T)$ is diagonal in the basis $\{\vert \lambda, N \rangle \}$,

$$\hat{C}(T)\vert \lambda, N \rangle = r_{\lambda}(N, T)\vert \lambda, N \rangle.$$ 

with eigenvalues:

$$r_{\lambda}(N, T) := r_0(N, T) \prod_{(i,j) \in \lambda} r_{N-i+j},$$

$$r_0(N, T) := \begin{cases} e^{\sum_{i=0}^{N-1} T_i} & \text{if } N > 0 \\ 1 & \text{if } N = 0 \\ e^{-\sum_{i=1}^{-N} T_{-i}} & \text{if } N < 0, \end{cases}$$
Lemma

Convolution actions multiply the coefficients in the Schur function expansions of $\tau_{C_\rho g}(N, t)$ and $\tau^{(2)}_{C_\rho \hat{g}C_{\tilde{\rho}}}(N, t, \tilde{t})$ by the diagonal factors $r_\lambda(N, T)$ and $r_\mu(N, \tilde{T})$.

$$\tau_{C_\rho g}(N, t) = \sum_\lambda r_\lambda(N, T)\pi_\lambda(g(H_+), N)s_\lambda(t),$$

$$\tau^{(2)}_{C_\rho gC_{\tilde{\rho}}}(N, t, \tilde{t}) = \sum_\lambda \sum_\mu r_\lambda(N, T)B_{\lambda, \mu}(g, N)r_\mu(N, \tilde{T})s_\lambda(t)s_\mu(\tilde{t}).$$

The Plücker coordinates for the modified Grassmannian elements $C_\rho g(H_+^N)$ and $C_\rho gC_{\tilde{\rho}}(w_\mu, N)$ are thus:

$$\pi_\lambda(C_\rho g(H_+), N) = r_\lambda(N, T)\pi_{N, g}(\lambda)$$

$$B_{\lambda, \mu}(C_\rho gC_{\tilde{\rho}}, N) = r_\lambda(N, T)B_{\lambda, \mu}(g, N))r_\mu(N, \tilde{T}).$$
1. New matrix models as $\tau$ functions. Example 1.

Example

$$
\rho_-(z) = \frac{1}{z} e^{\frac{1}{z}} = \sum_{i=0}^{\infty} \frac{z^{-i-1}}{i!}, \quad |z| \leq 1
$$

$$
\rho_+(z) = \frac{1}{1 - z} = \sum_{i=1}^{\infty} z^i \quad |z| > 1,
$$

$$
\rho_i = \begin{cases} 
\frac{1}{i!} & \text{if } i \geq 0 \\
1 & \text{if } i \leq -1,
\end{cases}
$$

$$
r_i = \begin{cases} 
\frac{1}{i} & \text{if } i \geq 1 \\
1 & \text{if } i \leq 0,
\end{cases}
$$

$$
r_\lambda(N) = \frac{1}{(\prod_{i=1}^{N-1} i!)(N)_{\lambda}} \quad \text{if } \ell(\lambda) \leq N
$$
New matrix models from old

Hermitian matrix integrals of the form

\[ Z_N(t) = \int_{M \in H^{N \times N}} d\mu(M) e^{tr \sum_{i=1}^{\infty} t_i M^i} \]

\[ = \prod_{a=1}^{N} \int_{\mathbb{R}} d\mu_0(x_a) e^{\sum_{i=1}^{\infty} t_i x_a^i} \Delta^2(X), \]

are KP-Toda \( \tau \)-functions. The Schur function expansion is

\[ Z_N(t) = \sum_{\ell(\lambda) \leq N} \pi_{N,d\mu}(\lambda) s_\lambda(t) \]

\[ \pi_{N,d\mu}(\lambda) = \prod_{a=1}^{N} \left( \int_{\mathbb{R}} d\mu_0(x_a) \right) \Delta^2(X) s_\lambda([X]) \]

\[ = (-1)^{1/2} N(N-1) N! \det(\mathcal{M}_{\lambda_i-i+j+N-1}) |_{1 \leq i,j \leq N} \]

\[ \mathcal{M}_{ij} := \int_{\mathbb{R}} d\mu_0(x) x^{i+j} \]
Now consider the **externally coupled** matrix model integral

\[
Z_{N,\text{ext}}(A) := \int_{M \in \mathbb{H}^{N \times N}} d\mu(M) e^{\text{tr}(AM)},
\]

where \(A \in \mathbb{H}^{N \times N}\) is a fixed \(N \times N\) Hermitian matrix. Applying the convolution symmetry of Example 1:

**Theorem**

Applying the convolution symmetry \(\tilde{C}_\rho\) to the \(\tau\)-function \(Z_N(t)\), where \(\rho_+(z)\) and \(\rho_-(z)\) are defined as in Example 1, and choosing the KP flow parameters as \(t = [A]\) gives, within a multiplicative constant, the externally coupled matrix integral

\[
\tilde{C}_\rho(Z_N)([A]) = \left( \prod_{i=1}^{N-1} i! \right)^{-1} Z_{N,\text{ext}}(A).
\]
Externally coupled two-matrix model integral

Itzykson-Zuber exponential coupled 2-matrix model

$$Z_N^{(2)}(t, \tilde{t}) = \int_{M_1 \in H^{N \times N}} d\mu(M_1) \int_{M_1 \in H^{N \times N}} d\tilde{\mu}(M_2) \ e^{\text{tr} \left( \sum_{i=1}^{\infty} \left( t_i M_1 + \tilde{t}_i M_2 + M_1 M_2 \right) \right)}$$

$$\propto \prod_{a=1}^{N} \left( \int_{\mathbb{R}} d\mu_0(x_a) \int_{\mathbb{R}} d\tilde{\mu}_0(y_a) \ e^{\sum_{i=1}^{\infty} (t_i x_a + \tilde{t}_i y_a + x_a y_a)} \right) \Delta(X) \Delta(Y)$$

**Theorem**

Applying the convolution symmetry $\tilde{C}_{\rho, \tilde{\rho}}$ to $Z_N^{(2)}$ and evaluating at the parameter values $t = [A], \tilde{t} = [B]$ gives the externally coupled matrix integral

$$\tilde{C}_{\rho, \tilde{\rho}}^{(2)}(Z_N^{(2)}([A], [B])) = Z_N^{(2), \rho, \tilde{\rho}}(A, B)$$
Convolution flows and the Q operator

The Q-operator

Choose an infinite sequence of constants \( \{q_j\}_{j \in \mathbb{Z}} \) with

\[ |q_j| > 1 \quad \text{for} \quad j > 0 \]

and define the infinite square matrix \( Q(q) \in \text{Mat}^{\mathbb{Z} \times \mathbb{Z}} \) having matrix elements

\[ Q_{ij} = (q_j)^i \]

\[ \Lambda Q = Q \; \text{diag}(q_i) \]

\[ \gamma_+(t)Q = Q \; C(T(q, t)) \]

\[ T_j(q, t) := \sum_{i=1}^{\infty} t_i(q_j)^i \]
The $Q$-operator (cont’d)

For suitably chosen values of $(q, \tilde{q})$ (see examples below), it is possible to make triangular decompositions

$$Q(q) = Q_-(q)Q_0(q)Q_+(q),$$

where $Q_0$, is of the form

$$Q_0(q) = \text{diag}(e^{\phi_j(q)}),$$

for a suitably defined infinite sequence

$$\phi(q) = \{\phi_j(q)\}, \quad j \in \mathbb{Z},$$

and $Q_\pm(q)$, $Q_\pm(\tilde{q})$ are invertible triangular matrices of the form

$$Q_\pm(q) = e^{A_\pm(q)}, \quad Q_\pm(\tilde{q}) = e^{A_\pm(\tilde{q})},$$

where $A^-(q)$ and $A^-(\tilde{q})$, $A^+(q)$, $A^+(\tilde{q})$ are, respectively, strictly lower ($-$) and strictly upper ($+$) triangular doubly infinite matrices.
Fermionic representation of the $Q$-operator

Introduce the fermionic vertex operators

\[
\hat{Q}_+(q) := e^{\sum_{i<j}^\infty A_{ij}^+(q)\psi_i\psi_j}, \quad \hat{Q}_-(q) := e^{\sum_{i>j}^\infty A_{ij}^-(q)\psi_i\psi_j}, \\
\hat{Q}_+(\tilde{q}) := e^{\sum_{i<j}^\infty A_{ij}^{-\dagger}(\tilde{q})\psi_i\psi_j}, \quad \hat{Q}_-(\tilde{q}) := e^{\sum_{i>j}^\infty A_{ij}^{\dagger}(\tilde{q})\psi_i\psi_j}, \\
\hat{C}(\phi(q)) := e^{\sum_{i\in\mathbb{Z}}^\infty \phi_i(q)K_i}, \quad \hat{C}(\phi(\tilde{q})) := e^{\sum_{i\in\mathbb{Z}}^\infty \phi_i(\tilde{q})K_i}.
\]

By the equivariance of the Plücker map, we then have

\[
\hat{\gamma}_+(t)\hat{Q}_-(q)\hat{C}(\phi(q))\hat{Q}_+(q) = \hat{Q}_-(q)\hat{C}(\phi(q))\hat{Q}_+(q)\hat{C}(T), \\
\hat{Q}_-(\tilde{q})\hat{C}(\phi(\tilde{q}))\hat{Q}_+(\tilde{q})\hat{\gamma}_-(\tilde{t}) = \hat{C}(\tilde{T})\hat{Q}_-(\tilde{q})\hat{C}(\phi(\tilde{q}))\hat{Q}_+(\tilde{q}).
\]
Convolution flows and $\tau$ functions

Introduce a new basis for the abelian algebra of convolution flow generators as follows:

$$K_j(q) := \sum_{i=-\infty}^{\infty} (q_i)^j K_i,$$

and define, correspondingly

$$\hat{C}_q(t) := e^{\sum_{i=1}^{\infty} t_i K_i(q)} = \hat{C}(T(q, t)),$$
$$\hat{C}_{\tilde{q}}(\tilde{t}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i K_i(\tilde{q})} = \hat{C}(T(\tilde{q}, \tilde{t})).$$
The fermionic representation of the tau function may be expressed in terms of the corresponding Convolution Symmetry flows as: follows

\[ \tau_g(q)(N,t) = r_0(N, \phi(q)) \langle N| \hat{Q}_+(q) \hat{C}_q(t) \hat{g} |N \rangle \]
\[ \tau^{(2)}_g(q,\tilde{q})(N,t,\tilde{t}) = r_0(N, \phi(q) + \phi(\tilde{q})) \langle N| \hat{Q}_+(q) \hat{C}_q(t) \hat{g} \hat{C}_\tilde{q}(\tilde{t}) \hat{Q}_-(\tilde{q}) |N \rangle, \]

where

\[ \hat{g}(q) := \hat{Q}_-(q) \hat{C}(\phi(q)) \hat{Q}_+(q) \hat{g} \]
\[ \hat{g}(q,\tilde{q}) := \hat{Q}_-(q) \hat{C}(\phi(q)) \hat{Q}_+(q) \hat{g} \hat{Q}_-(\tilde{q}) \hat{C}(\phi(\tilde{q})) \hat{Q}_+(\tilde{q}). \]
Triangular boundary operators $\hat{Q}(q), \hat{Q}(\bar{q})$ of Toeplitz type

**Example**

Let

$$q_j = e^{i\alpha q^{-j}}, \quad j \in \mathbb{Z}$$

where

$$q = e^{2\pi i \tau}, \quad \Im(\tau) > 0$$

and $\alpha = \alpha(q)$ is a real valued function of $q$. Then

$$Q(q)_{mn} = e^{im\alpha q^{-m-n}} = e^{im\alpha q^{-\frac{1}{2}m^2} q^{\frac{1}{2}(m-n)^2} e^{-\frac{1}{2}n^2}}$$

$$Q(q) = Q_0(q) \left( \sum_{m=-\infty}^{\infty} q^{\frac{m^2}{2}} a^{im\alpha \Lambda^m} \right) Q_0(q),$$

where

$$Q_0(q) = \text{diag}(q^{-\frac{1}{2}m^2})_{m \in \mathbb{Z}}$$
Triangular boundary operators $\hat{Q}(q), \hat{Q}(\bar{q})$ of Toeplitz type (cont’d)

**Example (cont’d)**

The infinite product formula for Jacobi theta functions implies

$$
\sum_{n=-\infty}^{\infty} q^{n^2} e^{i\alpha m} z^n = \nu(q) \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}} e^{i\alpha} z)(1 + q^{n-\frac{1}{2}} e^{-i\alpha} z^{-1})
$$

where

$$
\nu(q) = \prod_{n=1}^{\infty} (1 - q^n).
$$

Expressing the factors in the infinite product as

$$
1 + q^{n-\frac{1}{2}} e^{i\alpha} z = \exp \left( - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{i\alpha k} q^{k(n-\frac{1}{2})} z^k \right)
$$

and

$$
1 + q^{n-\frac{1}{2}} e^{-i\alpha} z = \exp \left( - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-i\alpha k} q^{k(n-\frac{1}{2})} z^{-k} \right)
$$
Triangular boundary operators $\hat{Q}(q), \hat{Q}(\bar{q})$ of Toeplitz type (cont’d)

Example (cont’d)

Replacing the complex parameter $z$ by the infinite shift matrix $\Lambda$, we obtain the factorization

$$Q(q) = \nu(q)Q_0(q)Q_-(\alpha, q)Q_+(\alpha, q)Q_0(q)$$

where

$$Q_{\pm}(\alpha, q) = \prod_{n=1}^{\infty} \gamma_{\pm}(m, \alpha, q)$$

are lower/upper triangular infinite Toeplitz matrices, and

$$\gamma_{\pm}(n, \alpha, q) := \exp \left( - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{i \alpha k} q^{k(n - \frac{1}{2})} \Lambda^{\pm k} \right).$$
Triangular boundary operators \( \hat{Q}(q), \hat{Q}(\bar{q}) \) of Toeplitz type (cont’d)

Example (cont’d)

The fermionic representation of this infinite matrix is therefore given by

\[
\hat{Q} = \nu(q)\hat{C}(\phi(q))\hat{Q}_-(\alpha, q)\hat{Q}_+(\alpha, q)\hat{C}(\phi(q))
\]

where

\[
\hat{Q}_\pm(\alpha, q) = \prod_{n=1}^{\infty} \hat{\gamma}_\pm(n, \alpha, q) = \exp \left( - \sum_{k=1}^{\infty} \frac{(-1)^k e^{i\alpha k} q^{\frac{k}{2}}}{k(1 - q^k)} J_{\pm k} \right)
\]

\[
\hat{\gamma}_\pm(n, \alpha, q) := \exp \left( - \sum_{k=1}^{\infty} \frac{(-1)^k e^{i\alpha k} q^{k(n-\frac{1}{2})}}{k} J_{\pm k} \right)
\]

\[
\phi(q) := \{\phi_j(q)\}, \quad \phi_j(q) = -i\pi \tau j^2, \quad j \in \mathbb{Z}.
\]
Triangular boundary operators $\hat{Q}(q), \hat{Q}(\tilde{q})$ of Toeplitz type (cont’d)

Example (cont’d)

The formula for the $\tau$ function therefore becomes

$$\tau_g(N, t) = r_0(N, \phi(q)) \langle N | \hat{Q}_+ (\alpha, q) \hat{C}(T) \hat{g}(\alpha, q) | N \rangle,$$

where

$$\hat{g}(\alpha, q) := \hat{Q}_+^{-1}(\alpha, q) \hat{Q}_-^{-1}(\alpha, q) \hat{C}^{-1}(\phi(q)) \hat{g},$$

$$T_j(q, t) := \sum_{k=1}^{\infty} t_j e^{i k \alpha} q^{-j k}.$$
Example (cont’d)

Similarly, we introduce a second pair \((\alpha(\tilde{q}), \tilde{q} = e^{2\pi i \tilde{\tau}})\) and define

\[
\hat{Q}_\pm(\tilde{\alpha}, \tilde{q}) := \hat{Q}_\pm^{-1}(\tilde{\alpha}, \tilde{q}).
\] (2.1)

Then the 2-Toda \(\tau\) function becomes

\[
\tau_g(N, t, \tilde{t}) = r_0(N, \phi(q) - \tilde{\phi}(\tilde{q})) \langle N \mid \hat{Q}_+ (\alpha, q) \hat{C}(T) \hat{g}(\alpha, \tilde{\alpha}, \tilde{q}) \hat{C}(\tilde{T}) \hat{Q}_- (\alpha, \tilde{q}) \mid N \rangle,
\]

where \(\hat{g}(\alpha, \tilde{\alpha}, q, \tilde{q}) =

\[
:= \hat{Q}_+^{-1}(\alpha, q) \hat{Q}_-^{-1}(\alpha, q) \hat{C}^{-1}(\phi(q)) \hat{g} \hat{C}^{-1}(\phi(\tilde{q})) \hat{Q}_+^{-1}(\tilde{\alpha}, \tilde{q}) \hat{Q}_-^{-1}(\tilde{\alpha}, \tilde{q}),
\]

\(\tilde{T}_j := \sum_{k=1}^{\infty} \tilde{t}_j e^{ik\tilde{\alpha}} \tilde{q}^{-jk}, \quad \phi_j(\tilde{q}) = -i\pi \tilde{\tau}j^2, \quad j \in \mathbb{Z}.
\)
Triangular boundary operators $\hat{Q}(q), \hat{Q}(\tilde{q})$ of Toeplitz type (cont’d)

In particular, choosing $\hat{g}$ so that

$$\hat{g}(\alpha, \tilde{\alpha}, q, \tilde{q}) = 1,$$

setting

$$\tilde{\alpha} = \alpha = \pi, \quad q = \tilde{q}, \quad t_i = \tilde{t}_i$$

and replacing $t_i$ by $\frac{1}{2} t_i$, we obtain the $q$-deformed partition function for plane partitions that was studied by Okounkov and Pandharipande, and by Nakatsu and Takahashi.

Other choices for the $q_j$’s give other “convolution flow” representations of various $\tau$ functions (cf. e.g. Wiegmann, Bettelheim, et al).
Background and related work

**Fermionic approach to \( \tau \) functions**


**Convolution symmetries, Matrix Models, \( \tau \) functions**


J. Harnad and A. Yu. Orlov, “Convolution symmetry flows and integrable hierarchies” (in preparation)

**Applications of convolution flows**

