Riemann-Hilbert Problems and new Soliton Equations

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It is my pleasure to congratulate Professor Jan Slawianowski for his 70-th birthday!
PLAN

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• On $N$-wave equations – $k = 1$
• New $N$-wave equations – $k \geq 2$
• mKdV equations related to simple Lie algebras
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Based on:


- V. S. Gerdjikov. *Derivative Nonlinear Schrödinger Equations with \( \mathbb{Z}_N \) and \( \mathbb{D}_N \)–Reductions.* Romanian Journal of Physics, 58, Nos. 5-6, 573-582 (2013).

- V. S. Gerdjikov, A. B. Yanovski *On soliton equations with \( \mathbb{Z}_h \)

RHP with canonical normalization

\[ \xi^+(x, t, \lambda) = \xi^-(x, t, \lambda)G(x, t, \lambda), \quad \lambda^k \in \mathbb{R}, \quad \lim_{\lambda \to \infty} \xi^+(x, t, \lambda) = 1, \]

\[ \xi^\pm(x, t, \lambda) \in \mathcal{G} \]

Consider particular type of dependence \( G(x, t, \lambda): \)

\[ i \frac{\partial G}{\partial x} - \lambda^k [J, G(x, t, \lambda)] = 0, \quad i \frac{\partial G}{\partial t} - \lambda^k [K, G(x, t, \lambda)] = 0. \]

where \( J \in \mathfrak{h} \subset \mathfrak{g}. \)

The canonical normalization of the RHP:

\[ \xi^\pm(x, t, \lambda) = \exp Q(x, t, \lambda), \quad Q(x, t, \lambda) = \sum_{k=1}^{\infty} Q_k(x, t) \lambda^{-k}. \]

where all \( Q_k(x, t) \in \mathfrak{g}. \) However,

\[ \mathcal{J}(x, t, \lambda) = \xi^\pm(x, t, \lambda)J \hat{\xi}^\pm(x, t, \lambda), \quad \mathcal{K}(x, t, \lambda) = \xi^\pm(x, t, \lambda)K \hat{\xi}^\pm(x, t, \lambda), \]
belong to the algebra $\mathfrak{g}$ for any $J$ and $K$ from $\mathfrak{g}$. If in addition $K$ also belongs to the Cartan subalgebra $\mathfrak{h}$, then

$$[\mathcal{J}(x, t, \lambda), \mathcal{K}(x, t, \lambda)] = 0.$$ 

Zakharov-Shabat theorem

**Theorem 1.** Let $\xi^\pm(x, t, \lambda)$ be solutions to the RHP whose sewing function depends on the auxiliary variables $x$ and $t$ as above. Then $\xi^\pm(x, t, \lambda)$ are fundamental solutions of the following set of differential operators:

$$L\xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial x} + U_s(x, t, \lambda) \xi^\pm(x, t, \lambda) - \lambda^k [J, \xi^\pm(x, t, \lambda)] = 0,$$

$$M\xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial t} + V(x, t, \lambda) \xi^\pm(x, t, \lambda) - \lambda^k [K, \xi^\pm(x, t, \lambda)] = 0.$$

**Proof.** Introduce the functions:

$$g^\pm(x, t, \lambda) = i \frac{\partial \xi^\pm}{\partial x} \hat{\xi}^\pm(x, t, \lambda) + \lambda^k \xi^\pm(x, t, \lambda) J \hat{\xi}^\pm(x, t, \lambda),$$

$$p^\pm(x, t, \lambda) = i \frac{\partial \xi^\pm}{\partial t} \hat{\xi}^\pm(x, t, \lambda) + \lambda^k \xi^\pm(x, t, \lambda) K \hat{\xi}^\pm(x, t, \lambda),$$
and using
\[ i \frac{\partial G}{\partial x} - \lambda^k [J, G(x, t, \lambda)] = 0, \quad i \frac{\partial G}{\partial t} - \lambda^k [K, G(x, t, \lambda)] = 0. \]
prove that
\[ g^+(x, t, \lambda) = g^-(x, t, \lambda), \quad p^+(x, t, \lambda) = p^-(x, t, \lambda), \]
which means that these functions are analytic functions of \( \lambda \) in the whole complex \( \lambda \)-plane. Next we find that:
\[ \lim_{\lambda \to \infty} g^+(x, t, \lambda) = \lambda^k J, \quad \lim_{\lambda \to \infty} p^+(x, t, \lambda) = \lambda^k K. \]
and make use of Liouville theorem to get
\[ g^+(x, t, \lambda) = g^-(x, t, \lambda) = \lambda^k J - \sum_{l=1}^{k} U_{s; l}(x, t) \lambda^{k-l}, \]
\[ p^+(x, t, \lambda) = p^-(x, t, \lambda) = \lambda^k K - \sum_{l=1}^{k} V_l(x, t) \lambda^{k-l}. \]
We shall see below that the coefficients $U_l(x, t)$ and $V_l(x, t)$ can be expressed in terms of the asymptotic coefficients $Q_s$ of $\xi^{\pm}(x, t, \lambda)$.

Now remember the definition of $g^+(x, t, \lambda)$

$$g^\pm(x, t, \lambda) = i \frac{\partial \hat{\xi}^\pm}{\partial x} \hat{\xi}^\pm(x, t, \lambda) + \lambda^k \xi^\pm(x, t, \lambda) J \hat{\xi}^\pm(x, t, \lambda)$$

$$= \lambda^k J - \sum_{l=1}^{k} U_{s; l}(x, t) \lambda^{k-l},$$

Multiply both sides by $\xi^{\pm}(x, t, \lambda)$ and move all the terms to the left:

$$i \frac{\partial \xi^\pm}{\partial x} + \sum_{l=1}^{k} U_l(x, t) \lambda^{k-l} \xi^\pm(x, t, \lambda) - \lambda^k [J, \xi^\pm(x, t, \lambda)] = 0,$$

i.e. $L \xi^{\pm}(x, t, \lambda) = 0$. $\square$

**Lemma 1.** The operators $L$ and $M$ commute

$$[L, M] = 0,$$
i.e. the following set of equations hold:

\[ i \frac{\partial U}{\partial t} - i \frac{\partial V}{\partial x} + [U(x, t, \lambda) - \lambda^k J, V(x, t, \lambda) - \lambda^k K] = 0. \]

where

\[ U(x, t, \lambda) = \sum_{l=1}^{k} U_l(x, t) \lambda^{k-l}, \quad V(x, t, \lambda) = \sum_{l=0}^{k} V_l(x, t) \lambda^{k-l}. \]

**Jets of order \( k \)**

How to parametrize \( U_s(x, t, \lambda) \) and \( V(x, t, \lambda) \)?

Use:

\[ \xi^\pm(x, t, \lambda) = \exp Q(x, t, \lambda), \quad Q(x, t, \lambda) = \sum_{k=1}^{\infty} Q_k(x, t) \lambda^{-k}. \]
and consider the jets of order $k$ of $\mathcal{J}(x, \lambda)$ and $\mathcal{K}(x, \lambda)$:

\[
\mathcal{J}(x, t, \lambda) \equiv \left( \lambda^k \xi^\pm(x, t, \lambda) J_l \hat{\xi}^\pm(x, t, \lambda) \right)_+ = \lambda^k J - U(x, t, \lambda),
\]

\[
\mathcal{K}(x, t, \lambda) \equiv \left( \lambda^k \xi^\pm(x, t, \lambda) K \hat{\xi}^\pm(x, t, \lambda) \right)_+ = \lambda^k K - V(x, t, \lambda).
\]

Express $U(x) \in \mathfrak{g}$ in terms of $Q_s(x)$:

\[
\mathcal{J}(x, t, \lambda) = J + \sum_{k=1}^{\infty} \frac{1}{k!} \text{ad}^k Q \mathcal{J}, \quad \mathcal{K}(x, t, \lambda) = K + \sum_{k=1}^{\infty} \frac{1}{k!} \text{ad}^k Q \mathcal{K},
\]

\[
\text{ad}_Q Z = [Q, Z], \quad \text{ad}^2_Q Z = [Q, [Q, Z]], \quad \ldots
\]

and therefore for $U_l$ we get:

\[
U_1(x, t) = -\text{ad}_Q J, \quad U_2(x, t) = -\text{ad}_Q^2 J - \frac{1}{2} \text{ad}_Q^3 J
\]

\[
U_3(x, t) = -\text{ad}_Q^3 J - \frac{1}{2} (\text{ad}_Q^2 \text{ad}_Q + \text{ad}_Q \text{ad}_Q^2) J - \frac{1}{6} \text{ad}_Q^4 J.
\]

and similar expressions for $V_l(x, t)$ with $J$ replaced by $K$. 
Reductions of polynomial bundles

a) \[ A \xi^+;^\dagger (x, t, \epsilon \lambda^*) \hat{A} = \hat{\xi}^-(x, t, \lambda), \quad AQ^\dagger (x, t, \epsilon \lambda^*) \hat{A} = -Q(x, t, \lambda), \]
b) \[ B \xi^+;^\ast (x, t, \epsilon \lambda^*) \hat{B} = \hat{\xi}^-(x, t, \lambda), \quad BQ^* (x, t, \epsilon \lambda^*) \hat{B} = Q(x, t, \lambda), \]
c) \[ C \xi^+;T (x, t, -\lambda) \hat{C} = \hat{\xi}^-(x, t, \lambda), \quad CQ^\dagger (x, t, -\lambda) \hat{C} = -Q(x, t, \lambda), \]

where \( \epsilon^2 = 1 \) and \( A, B \) and \( C \) are elements of the group \( G \) such that \( A^2 = B^2 = C^2 = 1 \). As for the \( \mathbb{Z}_N \)-reductions we may have:

\[ D \xi^\pm (x, t, \omega \lambda) \hat{D} = \xi^\pm (x, t, \lambda), \quad DQ(x, t, \omega \lambda) \hat{D} = Q(x, t, \lambda), \]

where \( \omega^N = 1 \) and \( D^N = 1 \).
On $N$-wave equations $- k = 1$

Lax representation involves two Lax operators linear in $\lambda$:

$$L\xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial x} + [J, Q(x, t)]\xi^\pm(x, t, \lambda) - \lambda [J, \xi^\pm(x, t, \lambda)] = 0,$$

$$M\xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial t} + [K, Q(x, t)]\xi^\pm(x, t, \lambda) - \lambda [K, \xi^\pm(x, t, \lambda)] = 0.$$

The corresponding equations take the form:

$$i \left[ J, \frac{\partial Q}{\partial t} \right] - i \left[ K, \frac{\partial Q}{\partial x} \right] - [[J, Q], [K, Q(x, t)]] = 0$$

$$Q(x, t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \quad J = \text{diag} (a_1, a_2, a_3),$$

$$K = \text{diag} (b_1, b_2, b_3),$$
Then the 3-wave equations take the form:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2} \frac{\partial u_1}{\partial x} + \kappa \epsilon_1 \epsilon_2 u_2^* u_3 &= 0, \\
\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3} \frac{\partial u_2}{\partial x} + \kappa \epsilon_1 u_1^* u_3 &= 0, \\
\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3} \frac{\partial u_3}{\partial x} + \kappa \epsilon_2 u_1^* u_2^* &= 0,
\end{align*}
\]

where

\[
\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2).
\]

**New 3-wave equations - \( k \geq 2 \)**

Let \( g = sl(3) \) and

\[
Q_1(x, t) = \begin{pmatrix} 0 & u_1 & u_3 \\
-v_1 & 0 & u_2 \\
-v_3 & -v_2 & 0 \end{pmatrix}, \quad Q_2(x, t) = \begin{pmatrix} q_{11} & w_1 & w_3 \\
-z_1 & q_{22} & w_2 \\
-z_3 & -z_2 & q_{33} \end{pmatrix},
\]
Fix up \( k = 2 \). Then the Lax pair becomes

\[
L \xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial x} + U(x, t, \lambda) \xi^\pm(x, t, \lambda) - \lambda^2 [J, \xi^\pm(x, t, \lambda)] = 0,
\]

\[
M \xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial t} + V(x, t, \lambda) \xi^\pm(x, t, \lambda) - \lambda^2 [K, \xi^\pm(x, t, \lambda)] = 0,
\]

where

\[
U \equiv U_2 + \lambda U_1 = \left( [J, Q_2(x)] - \frac{1}{2} [[J, Q_1], Q_1(x)] \right) + \lambda [J, Q_1],
\]

\[
V \equiv V_2 + \lambda V_1 = \left( [K, Q_2(x)] - \frac{1}{2} [[K, Q_1], Q_1(x)] \right) + \lambda [K, Q_1].
\]

Impose a \( \mathbb{Z}_2 \)-reduction of type a) with \( A = \text{diag} (1, \epsilon, 1) \), \( \epsilon^2 = 1 \). Thus \( Q_1 \) and \( Q_2 \) get reduced into:

\[
Q_1 = \begin{pmatrix}
0 & u_1 & 0 \\
\epsilon u_1^* & 0 & u_2 \\
0 & \epsilon u_2^* & 0
\end{pmatrix}, \quad Q_2 = \begin{pmatrix}
0 & 0 & w_3 \\
0 & 0 & 0 \\
w_3^* & 0 & 0
\end{pmatrix},
\]
and we obtain new type of integrable 3-wave equations:

\[ i(a_1 - a_2) \frac{\partial u_1}{\partial t} - i(b_1 - b_2) \frac{\partial u_1}{\partial x} + \epsilon \kappa u_2^* u_3 + \epsilon \frac{\kappa(a_1 - a_2)}{(a_1 - a_3)} u_1 |u_2|^2 = 0, \]

\[ i(a_2 - a_3) \frac{\partial u_2}{\partial t} - i(b_2 - b_3) \frac{\partial u_2}{\partial x} + \epsilon \kappa u_1^* u_3 - \epsilon \frac{\kappa(a_2 - a_3)}{(a_1 - a_3)} |u_1|^2 u_2 = 0, \]

\[ i(a_1 - a_3) \frac{\partial u_3}{\partial t} - i(b_1 - b_3) \frac{\partial u_3}{\partial x} - \frac{i \kappa}{a_1 - a_3} \frac{\partial (u_1 u_2)}{\partial x} + \epsilon \kappa \left( \frac{a_1 - a_2}{a_1 - a_3} |u_1|^2 + \frac{a_2 - a_3}{a_1 - a_3} |u_2|^2 \right) u_1 u_2 + \epsilon \kappa u_3 (|u_1|^2 - |u_2|^2) = 0, \]

where

\[ \kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2), \quad u_3 = w_3 + \frac{2a_2 - a_1 - a_3}{2(a_1 - a_3)} u_1 u_2. \]

The diagonal terms in the Lax representation are \( \lambda \)-independent.
Two of them read:

\[ i(a_1 - a_2) \frac{\partial |u_1|^2}{\partial t} - i(b_1 - b_2) \frac{\partial |u_1|^2}{\partial x} - \epsilon \kappa (u_1 u_2 u_3^* - u_1^* u_2 u_3) = 0, \]

\[ i(a_2 - a_3) \frac{\partial |u_2|^2}{\partial t} - i(b_2 - b_3) \frac{\partial |u_2|^2}{\partial x} - \epsilon \kappa (u_1 u_2 u_3^* - u_1^* u_2 u_3) = 0, \]

These relations are satisfied identically as a consequence of the NLEE.

**New types of 4-wave interactions**

The Lax pair for these new equations will be provided by:

\[ L \psi = i \frac{\partial \psi}{\partial x} + (U_2(x, t) + \lambda U_1(x, t) - \lambda^2 J) \psi(x, t, \lambda) = 0, \]

\[ M \psi = i \frac{\partial \psi}{\partial t} + (V_2(x, t) + \lambda V_1(x, t) - \lambda^2 K) \psi(x, t, \lambda) = 0, \]
where $U_j(x, t)$ and $V_j(x, t)$ are fast decaying smooth functions taking values in the Lie algebra $so(5)$

$$U_1(x, t) = [J, Q_1(x, t)], \quad U_2(x, t) = [J, Q_2(x, t)] - \frac{1}{2} \text{ad}_{Q_1}^2 J,$$

$$V_1(x, t) = [K, Q_1(x, t)], \quad V_2(x, t) = [K, Q_2(x, t)] - \frac{1}{2} \text{ad}_{Q_1}^2 K.$$

Here $\text{ad}_{Q_1} X \equiv [Q_1(x, t), X]$.

Assume $Q_1(x, t)$ and $Q_2(x, t)$ to be generic elements of $so(5)$:

$$Q_1(x, t) = \sum_{\alpha \in \Delta_+} (q^1_\alpha E_\alpha + p^1_\alpha E_{-\alpha}) + r^1_1 H_{e_1} + r^1_2 H_{e_2},$$

$$Q_2(x, t) = \sum_{\alpha \in \Delta_+} (q^2_\alpha E_\alpha + p^2_\alpha E_{-\alpha}) + r^2_1 H_{e_1} + r^2_2 H_{e_2},$$

$$J = a_1 H_{e_1} + a_2 H_{e_2} = \text{diag} (a_1, a_2, 0, -a_2, -a_1),$$

$$K = b_1 H_{e_1} + b_2 H_{e_2} = \text{diag} (b_1, b_2, 0, -b_2, -b_1),$$
Next we impose on $Q_1(x,t)$ and $Q_2(x,t)$ the natural reduction

$$B_0 U(x,t, \epsilon \lambda^*)^\dagger B_0^{-1} = U(x,t, \lambda), \quad B_0 = \text{diag} (1, \epsilon, 1, \epsilon, 1), \quad \epsilon^2 = 1.$$  

As a result:

$$B_0 (\chi^+(x,t, \epsilon \lambda^*))^\dagger B_0^{-1} = (\chi^-(x,t, \lambda))^{-1}, \quad B_0 (T(t, \epsilon \lambda^*))^\dagger B_0^{-1} = (T(t, \lambda))^{-1},$$

which provide $p_1^\alpha = \epsilon (q_1^\alpha)^*, \quad p_2^\alpha = \epsilon (q_2^\alpha)^*$. Then the Lax representation will be a (rather complicated) system of 8 NLEE for the 8 independent matrix elements $q_1^\alpha$ and $q_2^\alpha$.

However we can impose additional $\mathbb{Z}_2$ reduction condition

$$D \xi^\pm (x,t,-\lambda) \hat{D} = \xi^\pm (x,t,\lambda), \quad D Q(x,t,-\lambda) \hat{D} = Q(x,t,\lambda),$$

$$D = \text{diag} (1, -1, 1, -1, 1)$$
\[ Q_1(x, t) = u_1 E_{e_1 - e_2} + u_2 E_{e_2} + u_3 E_{e_1 + e_2} + v_1 E_{-e_1 + e_2} + v_2 E_{-e_2} + v_3 E_{-e_1 - e_2} \]
\[ = \begin{pmatrix} 0 & u_1 & 0 & u_3 & 0 \\ v_1 & 0 & u_2 & 0 & u_3 \\ 0 & v_2 & 0 & u_2 & 0 \\ v_3 & 0 & v_2 & 0 & u_1 \\ 0 & v_3 & 0 & v_1 & 0 \end{pmatrix}, \]
\[ Q_2(x, t) = u_4 E_{e_1} + v_4 E_{e_1} + w_1 H_{e_1} + w_2 H_{e_2} \]
\[ = \begin{pmatrix} w_1 & 0 & u_4 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ w_4 & 0 & 0 & 0 & u_4 \\ 0 & 0 & 0 & -w_2 & 0 \\ 0 & 0 & -v_4 & 0 & -w_1 \end{pmatrix}, \]
\[ J = a_1 H_{e_1} + a_2 H_{e_2} = \text{diag} (a_1, a_2, 0, -a_2, -a_1), \]
\[ K = b_1 H_{e_1} + b_2 H_{e_2} = \text{diag} (b_1, b_2, 0, -b_2, -b_1), \]

Combining both reductions for the matrix elements of \( Q_j(x, t) \) we have:

\[ v_1 = \epsilon u_1^*, \quad v_2 = \epsilon u_2^*, \quad v_3 = \epsilon u_3^*, \quad v_4 = u_4^*, \]
The commutativity condition for the Lax pair

\[ i \left( \frac{\partial V_2}{\partial x} + \lambda \frac{\partial V_1}{\partial x} \right) - i \left( \frac{\partial U_2}{\partial t} + \lambda \frac{\partial U_1}{\partial t} \right) + [U_2 + \lambda U_1 - \lambda^2 J, V_2 + \lambda V_1 - \lambda^2 K] = 0 \]

must hold identically with respect to \( \lambda \). The terms proportional to \( \lambda^4 \), \( \lambda^3 \) and \( \lambda^2 \) vanish identically. The term proportional to \( \lambda \) and the \( \lambda \)-independent term vanish provided \( Q_i \) satisfy the NLEE:

\[
\begin{align*}
  i \frac{\partial V_1}{\partial t} - i \frac{\partial U_1}{\partial t} + [U_2, V_1] + [U_1, V_1] &= 0, \\
  i \frac{\partial V_2}{\partial t} - i \frac{\partial U_2}{\partial t} + [U_2, V_2] &= 0.
\end{align*}
\]
In components the corresponding NLEE:

\[-2i(a_1 - a_2) \frac{\partial u_1}{\partial t} + 2i(b_1 - b_2) \frac{\partial u_1}{\partial x} + \kappa \epsilon u_2^* (\epsilon u_2^* u_3 - u_1 u_2 - 2u) = 0, \]

\[-2ia_2 \frac{\partial u_2}{\partial t} + 2ib_2 \frac{\partial u_2}{\partial x} - \kappa (u_2 \epsilon (|u_3|^2 - |u_1|^2) + 2u_3 u_4^* + 2\epsilon u_1^* u_4) = 0, \]

\[-2i(a_1 + a_2) \frac{\partial u_3}{\partial t} + 2i(b_1 + b_2) \frac{\partial u_3}{\partial x} + \kappa u_2 (\epsilon u_2^* u_3 - u_1 u_2 + 2u_4) = 0, \]

\[-2ia_1 \frac{\partial u_4}{\partial t} + 2ib_1 \frac{\partial u_4}{\partial x} + i \frac{\partial}{\partial t} (-(2a_2 - a_1)u_1 u_2 + (2a_2 + a_1)\epsilon u_2^* u_3)
+ i(2b_2 - b_1) \frac{\partial(u_1 u_2)}{\partial x} - i(2b_2 + b_1)\epsilon \frac{\partial(u_2^* u_3)}{\partial x} - \kappa (2\epsilon u_4 (|u_1|^2 - |u_3|^2)
+ \epsilon u_1 u_2 (|u_1|^2 + 3|u_3|^2) - u_3 u_2^* (3|u_1|^2 + |u_3|^2)) = 0. \]

Let us now introduce

\[U_4 = u_4 - \frac{1}{2a_1} ((a_1 - a_2)u_1 u_2 + (a_1 + a_2)\epsilon u_3 u_2^*). \]
As a result we get:

\[-2i(a_1 - a_2)\frac{\partial u_1}{\partial t} + 2i(b_1 - b_2)\frac{\partial u_1}{\partial x} - \frac{\kappa \epsilon}{a_1} u_2^*(2a_1 U_4 + \epsilon a_2 u_2^* u_3 + (2a_1 - a_2) u_1 u_2) = 0,\]

\[-2ia_2 \frac{\partial u_2}{\partial t} + 2ib_2 \frac{\partial u_2}{\partial x} - \frac{\kappa \epsilon}{a_1} u_2((2a_1 + a_2)|u_3|^2 - a_2|u_1|^2) - 2\kappa(u_3 U_4^* + \epsilon u_1^* U_4 + u_1^* u_2^* u_3) = 0,\]

\[-2i(a_1 + a_2)\frac{\partial u_3}{\partial t} + 2i(b_1 + b_2)\frac{\partial u_3}{\partial x} + \frac{\kappa}{a_1} u_2(\epsilon(2a_1 + a_2) u_2^* u_3 - a_2 u_1 u_2 + 2a_1 U_4) = 0,\]

\[-2ia_1 \frac{\partial U_4}{\partial t} + 2ib_1 \frac{\partial U_4}{\partial x} + \frac{\kappa}{a_1} \frac{\partial u_1 u_2}{\partial x} - \frac{\kappa \epsilon}{a_1} \frac{\partial u_2^* u_3}{\partial x} - \frac{\kappa}{a_1} (2\epsilon U_4(|u_1|^2 - |u_3|^2) + (\epsilon u_1^* u_2 - u_3^* u_2)((2a_1 - a_2)|u_1|^2 + (2a_1 + a_2)|u_3|^2)) = 0,\]
Soliton equations with $sl(n)$-series

\[ L\psi \equiv i \frac{\partial \psi}{\partial x} + U(x, t, \lambda)\psi = 0, \]
\[ M\psi \equiv i \frac{\partial \psi}{\partial t} + V(x, t, \lambda)\psi = \psi C(\lambda), \]

For the case of $\mathbb{Z}_N$-reduction (Mikhailov (1981)):

\[ C_1 U(x, t, \lambda)C_1^{-1} = U(x, t, \omega \lambda), \quad C_1 V(x, t, \lambda)C_1^{-1} = V(x, t, \omega \lambda), \]

where $C_1^N = 1$ is a Coxeter automorphism of the algebra $\mathfrak{sl}(N, \mathbb{C})$ and $\omega = \exp(2\pi i/N)$.

Let $\mathfrak{g} \simeq \mathfrak{sl}(N, \mathbb{C})$ and the group of reduction is $\mathbb{Z}_N$. The class of relevant NLEE may be considered as generalizations of the derivative NLS equations

\[ i \frac{\partial \psi_k}{\partial t} + \gamma \frac{\partial}{\partial x} \left( \cot \left( \frac{\pi k}{N} \right) \cdot \psi_k, x + i \sum_{p=1}^{N-1} \psi_p \psi_{k-p} \right) = 0, \]
\( k = 1, 2, \ldots, N - 1 \), where \( \gamma \) is a constant and the index \( k - p \) should be understood modulus \( N \) and \( \psi_0 = \psi_N = 0 \).

The automorphism \( \text{Ad}_{C_1} (\text{Ad}_{C_1}(Y)) \equiv C_1 Y C_1^{-1} \) for every \( Y \) from \( \mathfrak{g} \) defines a grading in the Lie algebra

\[
\mathfrak{sl}(N, \mathbb{C}) = \bigoplus_{k=0}^{N-1} \mathfrak{g}^{(k)},
\]

\[
J^{(k)} = \sum_{j=1}^{N} \omega^{kj} E_{j, j+s}, \quad C^{-1} J^{(k)} C = \omega^{-k} J^{(k)}.
\]

where \((E_{j,s})_{q,r} = \delta_{jq}\delta_{sr}\). Obviously

\[
\left[ J^{(k)}, J^{(m)}_l \right] = (\omega^{ms} - \omega^{kl}) J^{(k+m)}_{s+l}.
\]

Next choose \( U(x, t, \lambda) \) and \( V(x, t, \lambda) \) as follows:

\[
U(x, t, \lambda) = Q(x, t) - \lambda J, \quad Q(x, t) = \sum_{j=1}^{N-1} \psi_j(x, t) J^{(0)}_j, \quad J = aJ^{(1)}_0
\]
\[ V(x, t, \lambda) = V_3(x, t) + \lambda V_2(x, t) + \lambda^2 V_1(x, t) - \lambda^3 K, \]

where
\[
V_1(x, t) = \sum_{k=1}^{N} v_1^k(x, t) J_k^{(2)} , \quad V_2(x, t) = \sum_{l=1}^{N} v_2^l(x, t) J_l^{(1)} ,
\]
\[
V_3(x, t) = \sum_{j=1}^{N-1} v_3^j(x, t) J_j^{(0)} , \quad K = bJ_0^{(3)} .
\]

The constants \( a \) and \( b \) determine the dispersion law of the MKdV eqs.

The next step is to request that \([L, M] = 0\) identically with respect to \( \lambda \).

\[
v_1^1(x, t) = \frac{b}{a} (\omega^{2k} + \omega^k + 1) \psi_k , \quad k = 1, \ldots, N - 1 ,
\]
and $v^1_N = C(t)$ with $C(t)$ - arbitrary function of time. For

$$v^2_l(x, t) = \frac{b}{a^2} \sum_{j+k=l}^{N-1} \frac{\omega^{2l} + \omega^{2j+k} - \omega^k - 1}{1 - \omega^l} \psi_j \psi_k$$

$$+ i \frac{b}{a^2} \left( \frac{\omega^{2l} + \omega^l + 1}{1 - \omega^l} \right) \frac{\partial \psi_l}{\partial x} - \frac{C}{a} (\omega^l + 1) \psi_l,$$

for $l = 1, \ldots, N - 1$ and

$$v^2_N = -\frac{b}{a^2} \sum_{j+l=0}^{N-1} \left( \cos \frac{2\pi j}{N} + \frac{1}{2} \right) \psi_j \psi_l + D(t),$$

with $D(t)$ - another arbitrary function of time. And for

$$v^3_j = \frac{b}{a^3} \cot \left( \frac{\pi j}{N} \right) \sum_{k+l=j}^{N-1} \frac{\partial}{\partial x} (\psi_k \psi_l) + \frac{C}{a^2} \sum_{m+l=j}^{N-1} (\psi_m \psi_l)$$

$$+ \frac{b}{2a^3} \sum_{k+l=j}^{N-1} \frac{\cos \frac{\pi (k-l)}{N}}{\sin \frac{\pi j}{N}} \frac{\partial}{\partial x} (\psi_k \psi_l) - \frac{D}{a} \psi_j$$
\[
\begin{align*}
+ \frac{b}{a^3} & \sum_{l+m=j}^{N-1} \sum_{i+k=l}^{N-1} (\psi_i \psi_k \psi_m) + \frac{3b}{2a^3} \sum_{l+m=j}^{N-1} \cot \left( \frac{\pi l}{N} \right) \frac{\partial \psi_l}{\partial x} \psi_m \\
+ \frac{b}{a^3} & \sum_{l+m=j}^{N-1} \sum_{i+k=l}^{N-1} \frac{\sin \frac{\pi (j-2k)}{N} - \sin \frac{\pi (j-2m)}{N}}{\sin \frac{\pi j}{N}} (\psi_i \psi_k \psi_m) \\
- \frac{b}{4a^3} & \cot \left( \frac{\pi j}{N} \right) \sum_{l+m=j}^{N-1} \frac{\partial}{\partial x} (\psi_l \psi_m) + \frac{C}{a^2} \cot \left( \frac{\pi j}{N} \right) \frac{\partial \psi_j}{\partial x} \\
- \frac{b}{2a^3} & \sum_{l+m=j}^{N-1} \frac{\cos \frac{\pi (l-m)}{N}}{\sin \frac{\pi j}{N}} \frac{\partial}{\partial x} (\psi_l \psi_m) + \frac{b}{a^3} \left( \cot^2 \frac{\pi j}{N} - \frac{1}{4 \sin^2 \frac{\pi j}{N}} \right) \frac{\partial^2 \psi_j}{\partial x^2} \\
+ \frac{b}{a^3} & \sum_{k=1}^{N-1} \left( \cos \frac{2\pi k}{N} + \frac{1}{2} \right) (\psi_k \psi_{N-k} \psi_j)
\end{align*}
\]

where \( j \) is running from 1 to \( N-1 \). We choose \( C(t) = 0 \) and \( D(t) = 0 \).
In the end we get the following system of mKdV equations:

\[
\alpha \frac{\partial \psi_j}{\partial t} = \left( \cot^2 \frac{\pi j}{N} - \frac{1}{4 \sin^2 \frac{\pi j}{N}} \right) \frac{\partial^3 \psi_j}{\partial x^3} + \sum_{l+m=j}^{N-1} \sum_{i+k=l}^{N-1} \frac{\partial}{\partial x} (\psi_i \psi_k \psi_m)
\]

\[
+ \sum_{l+m=j}^{N-1} \sum_{i+k=l}^{N-1} \frac{\sin \left( \frac{\pi (j-2k)}{N} \right) - \sin \left( \frac{\pi (j-2m)}{N} \right)}{\sin \left( \frac{\pi j}{N} \right)} \frac{\partial}{\partial x} (\psi_i \psi_k \psi_m)
\]

\[
+ \sum_{k=1}^{N-1} \left( \cos \frac{2\pi k}{N} + \frac{1}{2} \right) \frac{\partial}{\partial x} (\psi_k \psi_{N-k} \psi_j) + \frac{3}{4} \cot \left( \frac{\pi j}{N} \right) \sum_{k+l=j}^{N-1} \frac{\partial^2}{\partial x^2} (\psi_k \psi_l)
\]

\[
+ \frac{3}{4} \sum_{k+l=j}^{N-1} \frac{\partial}{\partial x} \left( \cot \left( \frac{\pi l}{N} \right) \frac{\partial \psi_l}{\partial x} \psi_k + \cot \left( \frac{\pi k}{N} \right) \frac{\partial \psi_k}{\partial x} \psi_l \right)
\]

where \( \alpha = a^3/b \).
Additional Involutions and Examples

Along with the \( \mathbb{Z}_N \)-reduction we can introduce one of the following involutions (\( \mathbb{Z}_2 \)-reductions):

a) \( K_0^{-1} U^\dagger (x, t, \kappa_1(\lambda)) K_0 = U(x, t, \lambda), \quad \kappa_1(\lambda) = -\omega^{-1} \lambda^* \)

b) \( K_0^{-1} U^* (x, t, \kappa_1(\lambda)) K_0 = -U(x, t, \lambda), \quad \kappa_1(\lambda) = \omega^{-1} \lambda^* \)

c) \( U^T(x, t, -\lambda) = -U(x, t, \lambda), \)

where \( K_0^{-2} = 1 \). We choose

\[
K_0 = \sum_{k=1}^{N} E_{k,N-k+1}.
\]

The action of \( K_0 \) on the basis is as follows:

\[
K_0 \left( J^{(k)} \right)^{\dagger} K_0 = \omega^{k(s-1)} J^{(k)}, \quad K_0 \left( J^{(k)} \right)^* K_0 = \omega^{-k} J^{(k)}_{-s},
\]

from which there follow the reductions below.
An immediate consequences are the constraints on the potentials:

\[ K_0^{-1}Q^{\dagger}(x, t)K_0 = Q(x, t), \quad K_0^{-1}(J_0^{(1)})^{\dagger}K_0 = \omega^{-1}J_0^{(1)}, \]
\[ K_0^{-1}Q^*(x, t)K_0 = -Q(x, t), \quad K_0^{-1}(J_0^{(1)})^{*}K_0 = \omega^{-1}J_0^{(1)}, \]
\[ Q^T(x, t) = -Q(x, t), \quad (J_0^{(1)})^T = J_0^{(1)}. \]

More specifically there follows that each of the algebraic relations below:

a) \[ \psi^*_j(x, t) = \psi_j(x, t), \quad \alpha = \alpha^*; \]

b) \[ \psi^*_j(x, t) = -\psi_{N-j}(x, t), \quad \alpha = \alpha^*; \]

c) \[ \psi_j(x, t) = -\psi_{N-j}(x, t). \]

where \( j = 1, \ldots, N - 1 \), are compatible with the evolution of the MKdV eqs.

**Some particular cases**

Special examples of DNLS systems.
In the case of \( \mathfrak{sl}(2, \mathbb{C}) \) algebra we obtain the well-known MKdV equation
\[
\alpha \frac{\partial \psi_1}{\partial t} = -\frac{1}{4} \frac{\partial^3 \psi_1}{\partial x^3} - \frac{1}{2} \frac{\partial}{\partial x} (\psi_1^3).
\]

In the case of \( \mathfrak{sl}(3, \mathbb{C}) \) algebra we have the system of trivial equations \( \partial_t \psi_1 = 0 \) and \( \partial_t \psi_2 = 0 \). In the case of \( \mathfrak{sl}(4, \mathbb{C}) \) algebra we find:
\[
\alpha \frac{\partial \psi_1}{\partial t} = \frac{1}{2} \frac{\partial^3 \psi_1}{\partial x^3} + \frac{3}{2} \frac{\partial}{\partial x} \left( \frac{\partial \psi_2}{\partial x} \psi_3 \right) + \frac{3}{2} \frac{\partial}{\partial x} (\psi_1 \psi_2^2) + \frac{\partial}{\partial x} (\psi_3^3),
\]
\[
\alpha \frac{\partial \psi_2}{\partial t} = -\frac{1}{4} \frac{\partial^3 \psi_2}{\partial x^3} + \frac{3}{4} \frac{\partial^2}{\partial x^2} (\psi_1^2) - \frac{3}{4} \frac{\partial^2}{\partial x^2} (\psi_3^2)
+ 3 \frac{\partial}{\partial x} (\psi_1 \psi_2 \psi_3) - \frac{1}{2} \frac{\partial}{\partial x} (\psi_2^3), \quad (1)
\]
\[
\alpha \frac{\partial \psi_3}{\partial t} = \frac{1}{2} \frac{\partial^3 \psi_3}{\partial x^3} - \frac{3}{2} \frac{\partial}{\partial x} \left( \psi_1 \frac{\partial \psi_2}{\partial x} \right) + \frac{3}{2} \frac{\partial}{\partial x} (\psi_2^2 \psi_3) + \frac{\partial}{\partial x} (\psi_1^3).
\]

If we apply case a) we get the same set of MKdV equations with \( \psi_1, \psi_2 \) and \( \psi_3 \) purely real functions.
In the case b) we put $\psi_1 = -\psi_3^* = u$ and $\psi_2 = -\psi_2^* = iv$ and get:

$$\alpha \frac{\partial v}{\partial t} = - \frac{1}{4} \frac{\partial^3 v}{\partial x^3} + \frac{3}{4i} \frac{\partial^2}{\partial x^2} (u^2 - u^{*2}) - 3 \frac{\partial}{\partial x} (|u|^2 v) + \frac{1}{2} \frac{\partial}{\partial x} (v^3),$$

$$\alpha \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^3 u}{\partial x^3} - \frac{3}{2} \frac{\partial}{\partial x} \left( u^* \frac{\partial v}{\partial x} \right) - \frac{3}{2} \frac{\partial}{\partial x} (uv^2) - \frac{\partial}{\partial x} (u^*^3),$$

where $u$ is a complex function, but $v$ is a purely real function.

In the case c):

$$\alpha \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^3 u}{\partial x^3} - \frac{\partial}{\partial x} (u^3),$$

where $u$ is a complex function, we recover the well known MKdV equation. And finally in the case of $\mathfrak{sl}(6, \mathbb{C})$ algebra with $\mathbb{D}_6$-reduction in the case c) we find

$$\alpha \frac{\partial u}{\partial t} = 2 \frac{\partial^3 u}{\partial x^3} - 2\sqrt{3} \frac{\partial}{\partial x} \left( u \frac{\partial v}{\partial x} \right) - 6 \frac{\partial}{\partial x} (uv^2),$$

$$\alpha \frac{\partial v}{\partial t} = \sqrt{3} \frac{\partial^2}{\partial x^2} (u^2) - 6 \frac{\partial}{\partial x} (u^2 v),$$
where $u$ and $v$ are complex functions.

**MKdV and $so(8)$**

Normally with each simple Lie algebra one can associate just one MKdV eq.

The only exception is $s0(8)$ which allows a one-parameter family of MKdV equations. The reason is that only $so(8)$ has 3 as a double exponent!

$$
\partial_t q_1 = 2a \left[ \partial_x^3 q_1 - \sqrt{3} \partial_x (q_1 \partial_x q_2) \right] - \sqrt{3} \left[ (3a + b) \partial_x (q_4 \partial_x q_3) + (3a - b) \partial_x (q_3 \partial_x q_4) \right] \\
- 3 \partial_x \left[ q_1 \left( 2aq_2^2 + (a - b)q_3^2 + (a + b)q_4^2 \right) \right],
$$
\[ \partial_t q_2 = \sqrt{3} a \partial_x^2 q_1^2 + \frac{\sqrt{3}}{2} (a + b) \partial_x^2 q_3^2 + \frac{\sqrt{3}}{2} (a - b) \partial_x^2 q_4^2 \\
- 3 \partial_x \left[ q_2 \left( 2a q_1^2 + (a + b) q_3^2 + (a - b) q_4^2 \right) \right], \]

\[ \partial_t q_3 = -(a + b) \left[ \partial_x^3 q_3 - \sqrt{3} \partial_x (q_3 \partial_x q_2) \right] - \sqrt{3} \left[ (3a + b) \partial_x (q_4 \partial_x q_1) + 2b \partial_x (q_1 \partial_x q_4) \right] \\
+ 3 \partial_x \left[ q_3 \left( 2a q_4^2 + (a - b) q_1^2 + (a + b) q_2^2 \right) \right], \]

\[ \partial_t q_4 = -(a - b) \left[ \partial_x^3 q_4 - \sqrt{3} \partial_x (q_4 \partial_x q_2) \right] - \sqrt{3} \left[ (3a - b) \partial_x (q_3 \partial_x q_1) - 2b \partial_x (q_1 \partial_x q_3) \right] \\
+ 3 \partial_x \left[ q_4 \left( 2a q_3^2 + (a - b) q_2^2 + (a + b) q_1^2 \right) \right]. \]
Conclusions and open questions

- More classes of new integrable equations: i) higher rank simple Lie algebras; ii) different types of grading; iii) different power $k$ of the polynomials $U(x, t, \lambda)$ and $V(x, t, \lambda)$ and iv) different reductions of $U$ and $V$.

- These new NLEE must be Hamiltonian. View the jets $U(x, t, \lambda)$ and $V(x, t, \lambda)$ as elements of co-adjoint orbits of some Kac-Moody algebra.

- Apply Zakharov-Shabat dressing method for constructing their $N$-soliton solutions and study their interactions.

- Apply the above methods to twisted Kac-Moody algebras – work in progress
Thank you for your attention!