The Mylar Balloon: Alternative Parametrizations and Mathematica®

Vladimir Pulov¹
Mariana Hadjilazova,² Ivailo M. Mladenov²

¹Department of Physics, Technical University of Varna
²Institute of Biophysics, Bulgarian Academy of Science

Geometry, Integrability and Quantization
June 6-11, 2014
1. The Mylar
   Industrial and Geometrical
   Physical Construction
   Mathematical Model

2. Alternative Parametrizations
   Via the Elliptic Integrals
   Via the Weierstrassian Functions
   Mylar and Mathematica®

3. Geometrical Characteristics
   Radius and Thickness
   Surface Area and Volume
   Crimping Factor
The Physical Prototype of the Mylar Balloon
Mylar is a Trademark

- Mylar is extremely thin polyester film.
- Mylar is flexible and inelastic material.
- Mylar is having a great tensile stress.
The Mylar Sheets
Mylar is a Geometrical Figure

- Mylar (or Mylar balloon) is the name of a surface of revolution that resembles a slightly flattened sphere.
- The term Mylar was coined by (Paulsen, 1994) who first investigated the shape.
- Mylar is a surface that encloses maximum volume for a given directrice arclength.
Constructing the Mylar Balloon

- Take two circular disks made of Mylar.
- Sew the disks together along their boundaries.
- Inflate with either air or helium.
The Deflated Mylar

\[ a \]
First Geometrical Depiction  
(Paulsen, 1994)

- What is the shape of the inflated Mylar balloon?
- What is the radius of the inflated Mylar balloon?
- What is the thickness of the inflated Mylar balloon?
- What is the volume of the inflated Mylar balloon?
Mathematical Problem

Given a circular Mylar balloon what will be the shape of the balloon when it is fully inflated?
Preliminary Assumptions

- The deflated balloon lies in the $xy$-plane.
- The deflated balloon is centered at the origin.
- The deflated balloon has radius $a$.
- $Oz$ is the axis of revolution.
The Profile Curve

- The profile curve lies in the first quadrant $z = z(x), \ x \geq 0$.
- The axis of revolution is the $z$-axis.
- The bottom half of the Mylar is obtained by reflection of the upper half in the xy-plane.
The Supposed Profile of the Mylar
Calculus of Variations Problem

Find the profile curve

\[ z = z(x), \quad z(r) = 0, \quad x \geq 0 \]

by maximizing the volume

\[ V = 4\pi \int_0^r xz(x) \, dx \]

subject to the constraint

\[ \int_0^r \sqrt{1 + z'(x)^2} \, dx = a \]

and the transversality condition

\[ \lim_{x \to r^-} z'(x) = -\infty \]
The Mylar Balloon: Mathematical Model

The Euler-Lagrange Equation

\[
\frac{dz}{dx} = -\frac{x^2}{\sqrt{r^4 - x^4}}, \quad z(r) = 0, \quad 0 \leq x \leq r
\]
Further simplification of the last integral can be achieved by choosing appropriately the value of the elliptic modulus $k$ which in our considerations up to now is a free parameter. The basic relations among Jacobian functions hint about the possibility of taking $k \equiv 1/\sqrt{2}$ and we make just this choice as this gives the simplest result, namely

$$r \sqrt{2} \int_{0}^{\phi} \text{cn}^2(\phi, 1/\sqrt{2}) \, d\phi. \quad (11)$$

Using the definition $\cos u = \text{cn}(\phi, k)$ and that of the incomplete elliptic integrals $F(u, k), E(u, k)$ of the first, respectively second kind, it is a trivial matter to conclude that in our case we have

$$z(u) = r \sqrt{2} \left[ E(u, 1/\sqrt{2}) - \frac{1}{2} F(u, 1/\sqrt{2}) \right], \quad u \in [0, \pi/2]. \quad (12)$$

All this means that the profile curve (traced counter-clockwise) is

$$x(u) = r \cos u, \quad z(u) = r \sqrt{2} \left[ E(u, 1/\sqrt{2}) - \frac{1}{2} F(u, 1/\sqrt{2}) \right], \quad u \in [0, \pi/2],$$

while the surface of revolution generated by it, i.e. the mylar balloon surface $S$ can be represented in the form

$$x(u, v) = r \cos u \cos v, \quad y(u, v) = r \cos u \sin v, \quad z(u, v) = r \sqrt{2} \left[ E(u, 1/\sqrt{2}) - \frac{1}{2} F(u, 1/\sqrt{2}) \right], \quad u \in [-\pi/2, \pi/2], \quad v \in [0, 2\pi]. \quad (14)$$

Having the explicit parametrizations of the profile curve (13) and the surface of the mylar balloon (14) we now turn to study their geometry. Of principal importance is the relation between respective radii of deflated and inflated balloon. By (1) and (13) we have

$$\int_{\pi/2}^{0} \sqrt{x'(u)^2 + z'(u)^2} \, du = r \sqrt{2} \int_{\pi/2}^{0} \sqrt{1 - \frac{1}{2} \sin^2(u)} \, du = a. \quad (15)$$
The Euler-Lagrange Equation

\[ \frac{dx}{du} = \sqrt{r^4 - x^4} \]

\[ \frac{dz}{du} = -x^2, \quad 0 \leq x \leq r \]
The function \( x(u) \) is expressed by the Weierstrassian \( \wp(u) \)

\[
x(u) = c + \frac{f'(c)}{4} \left( \wp(u + C_1) - \frac{f''(c)}{24} \right)^{-1}
\]

where \( c \) is an arbitrary root of the polynomial

\[
f(\tau) = -\tau^4 + r^4
\]

with the invariants of \( \wp(u) \)

\[
g_2 = -r^4, \quad g_3 = 0
\]
The function $z(u)$ is expressed by

$$z(u) = 2c^4 J_1(u + C_1) - c^6 J_2(u + C_1) - c^2 u + C_2$$

$$J_1(u) = \frac{1}{\varphi'(\hat{u})} \left( 2\zeta(\hat{u}) u + \ln \frac{\sigma(u - \hat{u})}{\sigma(u + \hat{u})} \right)$$

$$J_2(u) = -\frac{1}{\varphi'^2(\hat{u})} \left( \varphi''(\hat{u}) J_1(u) + 2\varphi(\hat{u}) u + \zeta(u - \hat{u}) + \zeta(u + \hat{u}) \right)$$

where $\varphi(u)$, $\zeta(u)$, $\sigma(u)$ are the Weierstrassian functions and $\hat{u}$ denotes the argument of $\varphi(\cdot)$ which produces $\frac{f''(c)}{24}$.
Pseudo-Lemniscatic Weierstrassian Functions

\( (g_2 = -1, g_3 = 0) \)

\[
\begin{align*}
\wp''(u; -r^4, 0) &= r^4 \wp''(ru; -1, 0) \\
\wp'(u; -r^4, 0) &= r^3 \wp'(ru; -1, 0) \\
\wp(u; -r^4, 0) &= r^2 \wp(ru; -1, 0) \\
\zeta(u; -r^4, 0) &= r \zeta(ru; -1, 0) \\
\sigma(u; -r^4, 0) &= r^{-1} \sigma(ru; -1, 0)
\end{align*}
\]
The Profile of the Mylar
in Pseudo-Lemniscatic Weierstrassian Functions

On taking $c = r$ the solution is transformed to

$$x(u) = \frac{r(2\wp(ru; -1, 0) - 1)}{2\wp(ru; -1, 0) + 1}$$

$$z(u) = 2r^4 J_1(u + C_1) - r^6 J_2(u + C_1) - r^2 u + C_2$$

where $J_1(u), J_2(u)$ are expressed through the Pseudo-Lemniscatic Weierstrassian functions.
The Mylar Balloon
Mylar and Mathematica®

Mylar via Mathematica®

Figure 3. Two views of the Mylar balloon.

6. THE GEOMETRY OF THE MYLAR BALLOON.
Having the explicit parametrizations of the profile curve (19) and the surface of the Mylar balloon (20), we now turn to the study of their geometries. Of principal importance is the relation between the respective radii of the deflated and inflated balloons. From (8) and (19) we obtain the arclength (where we shorten $\text{sn}(u, 1/\sqrt{2})$ to $\text{sn} u, K(1/\sqrt{2})$ to $K, \text{etc.}$):

$$
\int_{0}^{K} \sqrt{x'(u)^2 + z'(u)^2} \, du = \int_{0}^{K} \sqrt{r^2 \text{sn}^2 u \, du} + \frac{r^2}{2} 4\text{cn}^4 u \, du = \int_{0}^{K} r \sqrt{\left(\text{sn}^2 u\right)\left(1 - \frac{1}{2}\text{sn}^2 u\right)} + \frac{1}{2} \left(1 - \text{sn}^2 u\right)^2 \, du = r \int_{0}^{K} \frac{1}{\sqrt{2}} \, du = r \sqrt{2} K = a.
$$
Geometrical Characteristics
Radius and Thickness

Radius  \[ r = \frac{\sqrt{2}}{K(1/\sqrt{2})} a \approx 0.7627a \]

Thickness  \[ \tau = 2\sqrt{2} \left[ E(1/\sqrt{2}) - \frac{1}{2} F(1/\sqrt{2}) \right] a \approx 0.9139a \]

Scale Invariance  \[ \frac{\tau}{2r} \approx 0.599 \]
Geometrical Characteristics
Surface Area and Volume

Surface Area
\[ A(S) = \pi^2 r^2 \]

Volume
\[ V = \frac{\pi \sqrt{2}}{3} K \left( \frac{1}{\sqrt{2}} \right) r^3 \]
Decrement of the Surface Area

\[ \frac{S_{\text{defl}}}{S_{\text{infl}}} = \frac{2\pi a^2}{\pi^2 r^2} \approx 1.0942 \]
Geometrical Characteristics

Crimping Factor

\[ C(x) = \frac{r^2}{x} \int_{0}^{x} \frac{dt}{\sqrt{r^4 - t^4}}, \quad 0 \leq x \leq r \]
Geometrical Characteristics
Crimping Factor

The Physical Crimping
References

References

References

Thank You!