0-Brane Matrix Dynamics for QCD purposes: Regge Trajectories

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Basic Idea

Matrix Dynamics

Quantum Dynamics

Angular momentum

Harmonic osc.

Rayleigh-Ritz Method and Spectrum
High energy Hadron-Hadron scatterings show two regimes:

1) Large momentum transfer: interaction among point-like substructures

2) Small momentum transfer: linear Regge trajectories are exchanged $\Rightarrow$ Motivation for String picture
Possible reconciliation of two regimes: Hadrons as bound-states of point-like Quarks and QCD flux-tubes (QCD-Strings).

Field theory anomalies of 2-dim string world-sheet

⇒ Lack of consistent QCD-Strings in 3+1 dims.
0-branes: Point-like objects to which strings end

Coordinates of $N$ 0-branes given by $N \times N$ hermitian matrices $\Rightarrow$ Strings’ dynamics encoded in off-diagonal elements of matrices
**Suggestion:** Modelling bound-state of Quarks and QCD-Strings by 0-brane matrix dynamics ⇒

Encouraging features [Fatollahi, EPL 53, 56, EPJC 19, 27, 17]:

1) Linear potential between static 0-branes
2) Regge behavior in scattering amplitudes
3) Whiteness of 0-branes’ c.m. w.r.t U(1) gauge fields on matrix space
**Q:** Do linear Regge trajectories emerge from 0-brane matrix dyn.? ⇒

Check Energy spectrum vs. Ang. Mom.

**Q:** Advantage of QM to world-sheet theory?

**A:** Absence of anomalies in QM of finite matrices

⇒ Possible consistent theory in 3+1 dims.
Q: Relevance of Matrix Coordinates to QCD physics?

A: 1) Special relativity lesson: 4-vector photon fields live in 4-vector space-time coordinates

2) SUSY lesson: anti-commuting coordinates represent fermion content

Matrix[YM] Fields $\leftrightarrow$ Matrix Coordinates $\rightarrow$

Who knows about the exact nature of Coordinates inside a proton?
Dynamics of $N$ 0-branes given by $U(N)$ YM theory dimensionally reduced to $0 + 1$ dimensions:

$$L = m_0 \text{Tr} \left( \frac{1}{2} (D_t X_i)^2 + \frac{1}{4 l_s^2} [X_i, X_j]^2 \right)$$

$i, j = 1, \ldots, d$, \quad $D_t = \partial_t - i [A_0, ]$

$X$: $N \times N$ hermitian matrices, \quad $X_i = x_{i a} T_a$

\(l_s\): string length, \quad $m_0 = (g_s l_s)^{-1}$

$m_0 \gg l_s^{-1}$. 
Theory is invariant under the gauge symmetry

\[ \vec{X} \rightarrow \vec{X}' = U\vec{X}U^\dagger, \]

\[ A_0 \rightarrow A'_0 = UA_0U^\dagger + iU\partial_t U^\dagger, \quad (1) \]

\( U \): arbitrary time-dependent \( N \times N \) unitary matrix

\[ D_t\vec{X} \rightarrow D'_t\vec{X}' = U(D_t\vec{X})U^\dagger, \]

\[ D_tD_t\vec{X} \rightarrow D'_tD'_t\vec{X}' = U(D_tD_t\vec{X})U^\dagger. \quad (2) \]
For each direction there are $N^2$ variables $\Rightarrow$

Extra $N^2 - N$ degrees of freedom represent dynamics of strings stretched between $N$ 0-branes.

c.m. is represented by trace of $X$ matrices.

QM of off-diagonal elements of matrices causes the interaction among 0-branes.
Example: quantum fluctuations of off-diagonal elements for classically static 0-branes ⇒ linear potential between 0-branes, just like QCD-string picture [Fatollahi, EPL 53]

Canonical momenta:

\[ P_i = \frac{\partial L}{\partial X_i} = m_0 D_t X_i \]  \hspace{1cm} (3)

Hamiltonian:

\[ H = \text{Tr} \left( \frac{P_i^2}{2 m_0} - \frac{m_0}{4 l_s^2} [X_i, X_j]^2 \right). \]  \hspace{1cm} (4)
As the time-derivative of the dynamical variable $A_0$ is absent, its equation of motion introduces constraints, the so-called Gauss’s law

$$G_a := \sum_i [X_i, P_i]_a = i \sum_{i,b,c} f_{abc} x_{i,b} p_{i,c} \equiv 0.$$ 

Pair of 0-branes ($N = 2$) in 2 dim ($d = 2$)
Possible decomposition: $SU(2) \times \Lambda \times SO(2)$ [Kares, NPB 689]

$X_{i\,a} = (\Psi)^a_{\,b}(\Lambda)^b_{\,j}(\eta)^j_{\,i}$

Matrix $\Psi$: SU(2) group element $\Rightarrow$ Gauge transformations of variable $X_{i\,a}$ are captured by $\Psi$ through ordinary gauge group left multiplications.

Parameterizing SU(2) by three Euler angles:

$\Psi = R_z(\alpha)R_x(\gamma)R_z(\beta)$,

$R_a$: rotation matrix about the ath axis.
Matrix $\eta$: $\text{SO}(2)$ group element parameterized by angle $\phi \Rightarrow$ capturing effect of rotation in 2-dim space

Remaining degrees: matrix $\Lambda$

$$
\Lambda = \begin{pmatrix}
    r \cos \theta & 0 \\
    0 & r \sin \theta \\
    0 & 0 \\
    0 & 0
\end{pmatrix}
$$
SU(2) pure gauge degrees: $\alpha, \beta, \gamma \Rightarrow \text{gauging away}$

by the Gauss law constraint:

\[
G_1 = \sin \alpha \cot \gamma \ p_\alpha - \sin \alpha \csc \gamma \ p_\beta - \cos \alpha \ p_\gamma = 0
\]

\[
G_2 = \cos \alpha \cot \gamma \ p_\alpha - \cos \alpha \csc \gamma \ p_\beta + \sin \alpha \ p_\gamma = 0
\]

\[
G_3 = -p_\alpha = 0
\]

$p_\alpha, \ p_\beta, \ p_\gamma$: canonical momenta

$p_\alpha = p_\beta = p_\gamma \equiv 0$. 
After imposing constraints, Hamiltonian is [Kares, NPB 689]:

\[
H = \frac{1}{2\mu} \left( p_r^2 + \frac{p_{\theta}^2}{r^2} + \frac{p_{\phi}^2}{r^2 \cos^2(2\theta)} \right) + \frac{\mu}{8} r^4 \sin^2(2\theta)
\]

\(\mu = m_0/2\): reduced mass of relative motion

\(p_{\phi}\): constant motion
Quantum theory

\[ p_\alpha \rightarrow -i \frac{\partial}{\partial \alpha}, \quad p_\beta \rightarrow -i \frac{\partial}{\partial \beta}, \quad p_\gamma \rightarrow -i \frac{\partial}{\partial \gamma} \]

Wave-function: independent of pure-gauge degrees
(as expected!)

Laplacian

\[ \nabla^2 \equiv \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j) \]
\[ H = -\frac{1}{2\mu} \left( \frac{1}{r^5} \partial_r \left( r^5 \partial_r \right) + \frac{1}{r^2} \nabla^2_\Omega \right) + \frac{\mu}{8} r^4 \sin^2(2\theta), \]

\[ \nabla^2_\Omega = \frac{1}{\sin(4\theta)} \partial_{\theta} \left( \sin(4\theta) \partial_{\theta} \right) + \frac{\partial^2_{\phi}}{\cos^2(2\theta)}. \]

Using scaling \( \psi \to r^{-3/2} \psi \)

\[ H = -\frac{1}{2\mu} \left( \frac{1}{r^2} \partial_r \left( r^2 \partial_r \right) + \frac{1}{r^2} \left( \nabla^2_\Omega - \frac{15}{4} \right) \right) + \frac{\mu}{8} r^4 \sin^2(2\theta), \]
\[ \nabla^2_{\Omega} \mathcal{Y}_\lambda(\theta, \phi) = \lambda \mathcal{Y}_\lambda(\theta, \phi) \]

\[ \mathcal{Y}_\lambda(\theta, \phi) = g_\lambda(\theta) \frac{e^{im_z\phi}}{\sqrt{2\pi}} \]

\( m_z: 0, \pm 2, \pm 4, \cdots \) (due to \( Z_2 \) sym)

New variable: \( x = \cos(4\theta), 0 \leq \theta \leq \pi/4 \)

\[ \frac{d}{dx} \left( (1 - x^2) \frac{dg_\lambda}{dx} \right) - \frac{m^2}{2(1 + x)} g_\lambda(x) = \frac{\lambda}{16} g_\lambda(x). \]
\[(1 - x^2)Q''(x) + (m - (m + 2)x)Q'(x)\]

\[- \left( \lambda + \frac{m(m + 2)}{4} \right) Q(x) = 0,\]

Solution: Jacobi polynomials of order

\[n = l - m \geq 0, \ P_n^{(0,m)}(x)\]

\[
\lambda = -16(l-m/2)(l-m/2+1), \quad m \leq l = 0, 1, \ldots ,
\]
Normalized Ang. Mom. eigenfunctions

\[ \mathcal{Y}_l^m(\theta, \phi) = \sqrt{\frac{2l - m + 1}{2^{m+1}}} (1 + \cos(4\theta))^{m/2} \mathcal{P}_{l-m}^{(0,m)}(\cos(4\theta)) \]

Recurrence relations:

\[ \frac{2(l+1)(l-m+1)}{(2l-m+1)(2l-m+2)} \mathcal{P}_{l-m+1}^{(0,m)}(x) + \frac{2l(l-m)}{(2l-m)(2l-m+2)} \mathcal{P}_{l-m}^{(0,m)}(x) + \frac{m^2}{(2l-m)(2l-m+2)} \mathcal{P}_{l-m}^{(0,m)}(x) = x \mathcal{P}_{l-m}^{(0,m)}(x) \]
Need a basis-function: H.O. (as usual!)

\[ \psi_{E,l,m}(r, \theta, \phi) = R_{E,l,m}(r) \mathcal{Y}_{l}^{m}(\theta, \phi) \]

Radial eq.

\[ -\frac{1}{2\mu} \left( R''_{E,l,m} - \frac{J_{l}^{m}(J_{l}^{m} + 1)}{r^{2}} R_{E,l,m} \right) + \frac{1}{2} \mu r^{2} R_{E,l,m} = E R_{E,l,m} \]

\[ J_{l}^{m} = 4l - 2m + 3/2. \]
$$R_{k,l,m}(r) = \sqrt{\frac{2k! \, \mu^{J^m_l+3/2}}{\Gamma(k + J^m_l + 3/2)}} \, r^{J^m_l} e^{-\mu r^2/2} \, L^{(J^m_l+1/2)}_k(\mu r^2)$$

$$k = 0, 1, 2, \cdots$$

$$E_{k,l,m} = 2k + J^m_l + 3/2 = 2k + 4l - 2m + 3$$
Basis function: 80 per Ang. Mom.

Rescalings:

\[ X_i \rightarrow g_s^{1/3} I_s X_i, \quad P_i \rightarrow g_s^{-1/3} I_s^{-1} P_i \]

\[ E \propto g_s^{1/3} I_s^{-1} \]
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<th>$E_3$</th>
<th>$E_4$</th>
<th>$E_5$</th>
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\[ E_1 = 3.474 [0.059] + 0.462 [0.002] m_z \]
\[ E_2 = 3.953 [0.031] + 0.500 [0.001] m_z \]
\[ E_3 = 4.632 [0.020] + 0.539 [0.001] m_z \]
\[ E_4 = 5.535 [0.027] + 0.579 [0.001] m_z \]
\[ E_5 = 6.754 [0.038] + 0.616 [0.001] m_z \]
\[ E_6 = 8.277 [0.047] + 0.654 [0.002] m_z \]

\[ m_z : 0, 2, 4, \ldots, 42 \]

Less than %2 error: straight-lines fit data!
Thank you on behalf of Matrix Coordinates!