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Bour surface companions in non-Euclidean space forms

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Introduction

- Minimal surfaces in 3-dimensional Euclidean space $\mathbb{R}^3$ isometric to rotational surfaces were first introduced by Bour [2] in 1862.
- see Güler [22], Güler, Yaylı and Hacısalihoğlu [23], Güler and Yaylı [24].
- also Ö zgür, Arslan and Murathan, [25].
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They are called Bour’s minimal surfaces $\mathcal{B}_m$ of value $m$. 
Furthermore, when $m$ is an integer greater than 1, $\mathcal{B}_m$ become algebraic, that is, there is an implicit polynomial equation satisfied by the three coordinates of $\mathcal{B}_m$, see also Gray [7], Nitsche [15], Whittemore [21].
Kobayashi [11] gave an analogous Weierstrass-type representation for conformal spacelike surfaces with mean curvature identically 0, called maximal surfaces, in 3-dimensional Minkowski space $\mathbb{R}^{2,1}$. 
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However, unlike the case of minimal surfaces in $\mathbb{R}^3$, maximal surfaces generally have singularities.
However, unlike the case of minimal surfaces in $\mathbb{R}^3$, maximal surfaces generally have singularities.

Details about singularities of maximal surfaces can be found in Fujimori et al [6], Umehara and Yamada [20].
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We remark that Magid [14] gave a Weierstrass-type representation for timelike surfaces with mean curvature identically 0, called timelike minimal surfaces, in $\mathbb{R}^{2,1}$, see also Inoguchi and Lee [10].
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Figure 1. Bour’s minimal surfaces of value 3 and 6 in $\mathbb{R}^3$. 

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Bour type surfaces
On the other hand, Lawson [12] showed that there is an isometric correspondence between constant mean curvature (CMC for short) surfaces in Riemannian space forms,
On the other hand, Lawson [12] showed that there is an isometric correspondence between constant mean curvature (CMC for short) surfaces in Riemannian space forms, and Palmer [16] showed that there is an analogous correspondence between spacelike CMC surfaces in Lorentzian space forms.
In particular, minimal surfaces in $\mathbb{R}^3$ correspond to CMC 1 surfaces in 3-dimensional hyperbolic space $\mathbb{H}^3$, and maximal surfaces in $\mathbb{R}^{2,1}$ correspond to CMC 1 surfaces in 3-dimensional de Sitter space $\mathbb{S}^{2,1}$. 
In particular, minimal surfaces in $\mathbb{R}^3$ correspond to CMC 1 surfaces in 3-dimensional hyperbolic space $\mathbb{H}^3$, and maximal surfaces in $\mathbb{R}^{2,1}$ correspond to CMC 1 surfaces in 3-dimensional de Sitter space $\mathbb{S}^{2,1}$.

Thus it is natural to expect existence of corresponding Weierstrass-type representations in these cases. Bryant [3] gave such a representation formula for CMC 1 surfaces in $\mathbb{H}^3$, and Umehara, Yamada [18] applied it.
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In particular, minimal surfaces in $\mathbb{R}^3$ correspond to CMC 1 surfaces in 3-dimensional hyperbolic space $\mathbb{H}^3$, and maximal surfaces in $\mathbb{R}^{2,1}$ correspond to CMC 1 surfaces in 3-dimensional de Sitter space $S^{2,1}$.

Thus it is natural to expect existence of corresponding Weierstrass-type representations in these cases. Bryant [3] gave such a representation formula for CMC 1 surfaces in $\mathbb{H}^3$, and Umehara, Yamada [18] applied it.

Similarly, Aiyama, Akutagawa [1] gave a representation formula for CMC 1 surfaces in $S^{2,1}$. 
However, analogues of Bour’s surfaces in other 3-dimensional space forms had not yet been studied.
In Sections 2 and 3 of this talk, in order to show that several maximal and timelike minimal Bour’s surfaces of value $m$ are algebraic, we review Weierstrass-type representations for maximal surfaces and timelike minimal surfaces in $\mathbb{R}^{2,1}$, and give explicit parametrizations for spacelike and timelike minimal Bour’s surfaces of value $m$. 
In Section 4, we introduce Bour type CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, and show several properties of those surfaces.
Introduction

In Section 4, we introduce Bour type CMC 1 surfaces in $\mathbb{H}^3$ and $S^{2,1}$, and show several properties of those surfaces.

Finally, in Section 5, we calculate the degrees, classes, implicit equations of the maximal and timelike minimal Bour’s surfaces of values 2, 3, 4 in $\mathbb{R}^{2,1}$ in terms of their coordinates.
We remark that in the cases of $\mathbb{H}^3$ and $S^{2,1}$, all surfaces are algebraic in some sense, because the Lorentz $(\mathbb{R}^{3,1})$ norm of all elements in $\mathbb{H}^3 \subset \mathbb{R}^{3,1}$ or $S^{2,1} \subset \mathbb{R}^{3,1}$ is constant.
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- What is the class of maximal and timelike minimal Bour’s surfaces of general value $m$ in $\mathbb{R}^{2,1}$?
However, we have the following remaining problems:

- What is the class of maximal and timelike minimal Bour’s surfaces of general value $m$ in $\mathbb{R}^{2,1}$?
- Are there any other implicit equations for CMC 1 Bour type surfaces? If there exist implicit equations, what are the corresponding degrees and classes?
Spacelike maximal Bour type surfaces

Let
\[ \mathbb{R}^{n,1} := \{ x = (x_1, \cdots, x_n, x_0)^t | x_i \in \mathbb{R} \}, \langle \cdot, \cdot \rangle \]
be the \((n + 1)\)-dimensional Lorentz-Minkowski (for short, Minkowski) space with Lorentz metric
\[ \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n - x_0 y_0. \]
Spacelike maximal Bour type surfaces

Then the 3-dimensional hyperbolic space $\mathbb{H}^3$ and 3-dimensional de Sitter space $S^{2,1}$ are defined as follows:

\[
\mathbb{H}^3 := \{ x \in \mathbb{R}^{3,1} | \langle x, x \rangle = -1, \ x_0 > 0 \} \cong \left\{ F\bar{F}^t | F \in SL_2\mathbb{C} \right\},
\]

\[
S^{2,1} := \{ x \in \mathbb{R}^{3,1} | \langle x, x \rangle = 1 \} \cong \left\{ F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{F}^t | F \in SL_2\mathbb{C} \right\}.
\]
A vector $x \in \mathbb{R}^{n,1}$ is called spacelike if $\langle x, x \rangle > 0$, timelike if $\langle x, x \rangle < 0$, and lightlike if $x \neq 0$ and $\langle x, x \rangle = 0$. 
Spacelike maximal Bour type surfaces

- A vector $x \in \mathbb{R}^{n,1}$ is called spacelike if $\langle x, x \rangle > 0$, timelike if $\langle x, x \rangle < 0$, and lightlike if $x \neq 0$ and $\langle x, x \rangle = 0$.

- A surface in $\mathbb{R}^{n,1}$ is called spacelike (resp. timelike, lightlike) if the induced metric on the tangent planes is a positive definite Riemannian (resp. Lorentzian, degenerate) metric.
Spacelike maximal Bour type surfaces in Minkowski 3-space

**Theorem (1)**

Let \( g, \omega \) be holomorphic functions defined on a simply connected open subset \( \mathcal{U} \subset \mathbb{C} \) such that \( \omega \) does not vanish on \( \mathcal{U} \). Then

\[
f(z) = \text{Re} \int \begin{pmatrix} (1 + g^2) \omega \\ i(1 - g^2) \omega \\ 2g \omega \end{pmatrix} \, dz
\]

is a spacelike conformal immersion with mean curvature identically 0 (i.e. spacelike conformal maximal surface). Conversely, any spacelike conformal maximal surface can be described in this manner.
Remark (1). A pair of a holomorphic function $g$ and a holomorphic function $\omega$, $(g, \omega)$ is called Weierstrass data for a maximal surface. In Section 4, we also call $(g, \omega)$ the Weierstrass data for CMC 1 surfaces in $\mathbb{H}^3$ and $S^{2,1}$. 
We call maximal surfaces $\mathcal{B}_m$ ($m \in \mathbb{Z}_{\geq 2} := \{n \in \mathbb{Z} | n \geq 2\}$) given by $(g, \omega) = (z, z^{m-2})$ the spacelike Bour’s maximal surfaces $\mathcal{B}_m$ of value $m$ (spacelike $\mathcal{B}_m$, for short).
Spacelike maximal Bour type surfaces

We call maximal surfaces \( \mathcal{B}_m \) (\( m \in \mathbb{Z}_{\geq 2} := \{ n \in \mathbb{Z} \mid n \geq 2 \} \)) given by \((g, \omega) = (z, z^{m-2})\) the spacelike Bour’s maximal surfaces \( \mathcal{B}_m \) of value \( m \) (spacelike \( \mathcal{B}_m \), for short).

Several properties of spacelike \( \mathcal{B}_m \) can be found in Güler [8].
Spacelike maximal Bour type surfaces in Minkowski 3-space

Timelike minimal Bour type surfaces in Minkowski 3-space

CMC 1 Bour type surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$

Degree and class of Bour type surfaces in $\mathbb{R}^{2,1}$

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Spacelike maximal Bour type surfaces

The parametrization of spacelike $\mathcal{B}_m(u, v)$ is

$$\text{Re} \left( \frac{1}{m-1} \sum_{k=0}^{m-1} \binom{m-1}{k} u^{m-1-k} (iv)^k + \frac{1}{m+1} \sum_{k=0}^{m+1} \binom{m+1}{k} u^{m+1-k} (iv)^k \right)$$

$$\frac{i}{m-1} \sum_{k=0}^{m-1} \binom{m-1}{k} u^{m-1-k} (iv)^k - \frac{i}{m+1} \sum_{k=0}^{m+1} \binom{m+1}{k} u^{m+1-k} (iv)^k$$

$$\frac{2}{m} \sum_{k=0}^{m} \binom{m}{k} u^{m-k} (iv)^k$$

(1)
Spacelike maximal Bour type surfaces

with Gauss map

\[ n = \left( \frac{2u}{u^2 + v^2 - 1}, \frac{2v}{u^2 + v^2 - 1}, \frac{u^2 + v^2 + 1}{u^2 + v^2 - 1} \right), \]

where \( z = u + iv \).
Next, we give the Weierstrass-type representation for timelike minimal surfaces in $\mathbb{R}^{2,1}$, which was obtained by M. Magid [14] (see also Inoguchi and Lee [10]).
Timelike minimal Bour type surfaces

Theorem (2)

Let $g_1(u)$, $\omega_1(u)$ (resp. $g_2(v)$, $\omega_2(v)$) be smooth functions depending on only $u$ (resp. $v$) on a connected orientable 2-manifold with local coordinates $u$, $v$. Then

$$\hat{f}(u, v) = \int \left( \begin{array}{c} 2g_1 \omega_1 \\ (1 - g_1^2) \omega_1 \\ - (1 + g_1^2) \omega_1 \end{array} \right) du + \int \left( \begin{array}{c} 2g_2 \omega_2 \\ (1 - g_2^2) \omega_2 \\ (1 + g_2^2) \omega_2 \end{array} \right) dv.$$ 

is a timelike surface with mean curvature identically 0 (i.e. timelike minimal surface). Conversely, any timelike minimal surface can be described in this manner.
The timelike minimal surfaces given by 
\((g_1(u), \omega_1(u)) = (u, u^{m-2}), (g_2(v), \omega_2(v)) = (v, v^{m-2})\) are called timelike Bour surfaces \(B_m\) of value \(m\) (timelike \(B_m\), for short) in \(\mathbb{R}^{2,1}\), where \(m \in \mathbb{Z}_{\geq 2}\).
The parametrization of timelike \( \mathcal{B}_m \) is

\[
\mathcal{B}_m(u, v) = \left(\frac{1}{m-1} (u^{m-1} + v^{m-1}) - \frac{1}{m+1} (u^{m+1} + v^{m+1})\right) - \frac{2}{m} (u^m + v^m)
\] 

\[
-\frac{1}{m-1} (u^{m-1} - v^{m-1}) - \frac{1}{m+1} (u^{m+1} - v^{m+1})
\]

(2)
Timelike minimal Bour type surfaces

with Gauss map

\[ n = \left( \frac{uv - 1}{1 + uv}, \frac{u + v}{1 + uv}, \frac{u - v}{1 + uv} \right). \]
Timelike minimal Bour type surfaces

Figure 2. Left two pictures: spacelike $\mathcal{B}_3$ and $\mathcal{B}_6$ in $\mathbb{R}^{2,1}$, right two pictures: timelike $\mathcal{B}_3$ and $\mathcal{B}_6$ in $\mathbb{R}^{2,1}$
In this section we consider CMC 1 surfaces in $\mathbb{H}^3$ and $S^{2,1}$. Here we identify elements in $\mathbb{H}^3$ and $S^{2,1}$ with $SL_2\mathbb{C}$ matrix forms as in Section 2.
In this section we consider CMC 1 surfaces in $\mathbb{H}^3$ and $S^{2,1}$. Here we identify elements in $\mathbb{H}^3$ and $S^{2,1}$ with $\text{SL}_2\mathbb{C}$ matrix forms as in Section 2.

In this setting Bryant [3] showed the following representation formula for CMC 1 surfaces in $\mathbb{H}^3$:
CMC 1 Bour type surfaces

Theorem (3)

Let $F \in SL_2 \mathbb{C}$ be a solution of the equation

$$dF = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega, \quad F|_{z=z_0} \in SL_2 \mathbb{C}$$

for some $z_0$ in a given domain, where $(g, \omega)$ is Weierstrass data. Then the surface $f = F \bar{F}^t$ is a conformal CMC 1 immersion into $\mathbb{H}^3$. Conversely, any conformal CMC 1 immersion in $\mathbb{H}^3$ can be described in this way. The metric of $f$ is $(1 + |g|^2)^2 |\omega|^2$. 
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Similarly, Aiyama and Akutagawa [1] showed the following Bryant-type representation formula for CMC 1 surfaces in $S^{2,1}$:
Theorem (4)

Let $\hat{F} \in \text{SL}_2\mathbb{C}$ be a solution of Equation (3), where $(g, \omega)$ is Weierstrass data. Then the surface $f = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{F}^t$ is a spacelike conformal CMC 1 immersion into $S_{2,1}$. Conversely, any spacelike conformal CMC 1 immersion in $S_{2,1}$ is described in this way. The metric of $f$ is $(1 - |g|^2)^2 |\omega|^2$. 
Note that, unlike in $\mathbb{H}^3$, CMC 1 surfaces in $\mathbb{S}^{2,1}$ generally have singularities. Their singularities have been investigated Fujimori et al [6], Umehara and Yamada [20].
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We call CMC 1 surfaces in $\mathbb{H}^3$ and $S_{2.1}$ given by the Weierstrass data $(g, \omega) = (z, z^{m-2})$ the Bour type CMC 1 cousins $\mathfrak{B}_m$ of value $m$ ($\mathfrak{B}_m$ cousin, for short).
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We now describe $F$ explicitly:
CMC 1 Bour type surfaces

Theorem (5)

Let $F(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \in \text{SL}_2\mathbb{C}$ be a solution of Equation (3) with $(g, \omega) = (z, z^{m-2}dz)$ and with initial condition $F(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. 
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Theorem (5)

(Cont.) Then

\[ a(z) = m^{1/m} \Gamma \left( \frac{m+1}{m} \right) z^{m-1/2} \text{Bessel I} \left( -\frac{m-1}{m}, \frac{2}{m} z^{m/2} \right), \]

\[ b(z) = -m^{1/m} \Gamma \left( \frac{m+1}{m} \right) z^{m+1/2} \text{Bessel I} \left( \frac{m+1}{m}, \frac{2}{m} z^{m/2} \right), \]

\[ c(z) = m^{-1/m} \Gamma \left( \frac{m-1}{m} \right) z^{m-1/2} \text{Bessel I} \left( \frac{m-1}{m}, \frac{2}{m} z^{m/2} \right), \]

\[ d(z) = -m^{-1/m} \Gamma \left( \frac{m-1}{m} \right) z^{m+1/2} \text{Bessel I} \left( -\frac{m+1}{m}, \frac{2}{m} z^{m/2} \right), \]
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Theorem (5)

(Cont.) where $\Gamma$ denotes the Gamma function and $Bessel I$ represents the modified Bessel function.

- The definition of $Bessel I$ can be found in standard textbooks, for example, see [9].
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Figure 3. Left two pictures: $\mathcal{B}_3$ cousin in $\mathbb{H}^3$, right two pictures: its dual cousin in $\mathbb{H}^3$ (in the Poincare ball model for $\mathbb{H}^3$)
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Figure 4. Left two pictures: $\mathcal{B}_6$ cousin in $\mathbb{H}^3$, right two pictures: its dual cousin in $\mathbb{H}^3$
CMC 1 Bour type surfaces

Proof.
Equation (3) gives

\[ X'' - \frac{\omega'}{\omega} X' - g' \omega X = 0, \quad (X = a(z), \ c(z)) \quad (5) \]

\[ Y'' - \frac{(g^2 \omega)'}{g^2 \omega} Y' - g' \omega Y = 0 \quad (Y = b(z), \ d(z)), \quad (6) \]

which are given by Umehara and K. Yamada [18]. Here we solve Equation (5).
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Proof. (cont.) Inserting \((g, \omega) = (z, z^{m-2})\) into Equation (5), we have

\[
X'' - \frac{m - 2}{z} X' - z^{m-2} X = 0. \quad (m \in \mathbb{Z}_{\geq 2})
\] (7)
Proof. (cont.)
We give two independent power series solutions of the differential equation (7) by the Frobenius method. The indicial equation at $z = 0$ is $\rho(\rho - 1) - (m - 2)\rho = 0$. So we see that the characteristic exponents of the equation (7) are 0 and $m - 1$. 
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Proof. (cont.)
Then we have a solution of the form

\[ z^{m-1} \sum_{p=0}^{\infty} a_p z^p, \]

where the coefficients \( a_p \) are inductively given by

\[
a_{mk+l} = 0 \quad (l = 0, \cdots, m),
\]

\[
a_{mk+m+1} = \frac{a_{m(k-1)+m-1}}{(m-2)k(mk + m - 1)}
\]

\[
= \frac{\Gamma\left(\frac{m-1}{m} + k\right)}{m^2 \Gamma\left(\frac{m-1}{m} + k + 1\right)} a_{m(k-1)+m-1} \quad (l \geq m + 1).
\]
Proof. (cont.)

Therefore we obtain a solution of the differential equation (7):

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma\left(\frac{m-1}{m} + k + 1\right)} \left(\frac{z^2}{m}\right)^{2k + \frac{m-1}{m}}
\]

\[
= z^{\frac{m-1}{2}} \text{Bessel I} \left(\frac{m-1}{m}, \frac{2}{m} z^2\right).
\]
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Proof. (cont.)

Similarly, we obtain another independent solution as

\[
z^{\frac{m-1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(-\frac{m-1}{m} + k + 1)} \left( \frac{z^{\frac{m}{2}}}{m} \right)^{2k - \frac{m-1}{m}}
\]

\[= z^{\frac{m-1}{2}} \text{Bessel I} \left( -\frac{m-1}{m}, \frac{2}{m} z^{\frac{m}{2}} \right) .\]
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Proof. (cont.)
So we have two independent solutions of Equation (5). Next, we find two independent solutions of Equation (6).
Proof. (cont.)
Inserting \((g, \omega) = (z, z^{m-2})\) into Equation (6), we have

\[
Y'' - \frac{m}{z} Y' - z^{m-2} Y = 0. \quad (m \in \mathbb{Z}_{\geq 2})
\]
Proof. (cont.)
Similarly to the way we solved Equation (5), we have two independent solutions

\[ z^{m+1 \over 2} \text{Bessel I} \left( {m + 1 \over m}, 2 {m \over m} z^{m \over 2} \right), \quad z^{m+1 \over 2} \text{Bessel I} \left( -{m + 1 \over m}, 2 {m \over m} z^{m \over 2} \right). \]
Proof. (cont.)
Using the initial conditions, we have the solution $F$ as in Equations (4).
**Remark (2).** If $F$ is a solution of Equation (3), the surface

\[ f^\# = (F^{-1})(F^{-1})^t \quad (\text{resp. } f^\# = (F^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (F^{-1})^t) \]

is also a CMC 1 surface in \( \mathbb{H}^3 \) (resp. \( S^{2,1} \)). This was proven by Umehara and Yamada [19] (resp. Lee [13]). The surface $f^\#$ is called the CMC 1 dual of $f$. 
CMC 1 Bour type surfaces

Using the explicit parametrization of the $\mathcal{B}_m$ cousin, we can easily show the following corollary, which implies the rotational symmetric property of the $\mathcal{B}_m$ cousins in $\mathbb{H}^3, S^{2,1}$. 
Corollary (1)

Let \( F(z) \in SL_2\mathbb{C} \) be the form as in Theorem 5 with complex coordinate \( z \). Then

\[
F(e^{i\frac{2\pi}{m}} \cdot z) = \begin{pmatrix} a(z) & \text{e}^{i\frac{2\pi}{m}} \cdot b(z) \\ \text{e}^{-i\frac{2\pi}{m}} \cdot c(z) & d(z) \end{pmatrix}.
\]
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Writing \( \mathcal{B}_m \) cousin in \( \mathbb{H}^3 \) or \( S^{2,1} \) as
\[
f(z) = (x_1(z), x_2(z), x_3(z), x_0(z))^t,
\]
given by Theorem 5, and setting
\[
f\left( e^{i \frac{2 \pi}{m} \cdot z} \right) = (\hat{x}_1(z), \hat{x}_2(z), \hat{x}_3(z), \hat{x}_0(z))^t.
\]
By Corollary (1), we have

\[
\begin{align*}
\hat{x}_1(z) &= \cos \left( \frac{2\pi}{m} \right) x_1(z) - \sin \left( \frac{2\pi}{m} \right) x_2(z), \\
\hat{x}_2(z) &= \sin \left( \frac{2\pi}{m} \right) x_1(z) + \cos \left( \frac{2\pi}{m} \right) x_2(z), \\
\hat{x}_3(z) &= x_3(z), \quad \hat{x}_0(z) = x_0(z),
\end{align*}
\]

that is, by rotating \( z \) by angle \( \frac{2\pi}{m} \), the first and second coordinates are also rotated by the same angle.
CMC 1 Bour type surfaces

So like in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$, $B_m$ has symmetry with respect to rotation by angle $\frac{2\pi}{m}$. Its dual $(B_m)^\#$ also has the same symmetry.
CMC 1 Bour type surfaces

In order to see CMC 1 surfaces in $\mathbb{H}^3$, we use a stereographic projection.
In order to see CMC 1 surfaces in $\mathbb{H}^3$, we use a stereographic projection.

Consider the map

$$\mathbb{H}^3 \ni (x_1, x_2, x_3, x_0)^t \mapsto \left( \frac{x_1}{1 + x_0}, \frac{x_2}{1 + x_0}, \frac{x_3}{1 + x_0} \right)^t \in \mathbb{B}^3,$$

where $\mathbb{B}^3$ denotes the 3-dimensional unit ball.
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where $\mathbb{B}^3$ denotes the 3-dimensional unit ball.

This is the Poincaré ball model for $\mathbb{H}^3$. 
CMC 1 Bour type surfaces

In order to show graphics of CMC 1 surfaces in $S^{2,1}$, the hollow ball model is used, see Fujimori [4] for example.
CMC 1 Bour type surfaces

In order to show graphics of CMC 1 surfaces in $S^{2,1}$, the hollow ball model is used, see Fujimori [4] for example.

Consider the map

$$S^{2,1} \ni (x_1, x_2, x_3, x_0)^t \mapsto \left( \frac{e^{\arctan(x_0)} \cdot x_1}{\sqrt{1 + x_0^2}}, \frac{e^{\arctan(x_0)} \cdot x_2}{\sqrt{1 + x_0^2}}, \frac{e^{\arctan(x_0)} \cdot x_3}{\sqrt{1 + x_0^2}} \right)^t \in \mathbb{B}^3_{(-\pi, \pi)},$$

where

$$\mathbb{B}^3_{(-\pi, \pi)} := \{(y_1, y_2, y_3)^t \in \mathbb{R}^3 \mid e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^\pi\}.$$
Figure 5. Left two pictures: $\mathcal{V}_3$ cousin in $S^{2,1}$, right two pictures: its dual cousin in $S^{2,1}$
CMC 1 Bour type surfaces

Figure 6. Left two pictures: $V_6$ cousin in $S^{2,1}$, right two pictures: its dual cousin in $S^{2,1}$
Degree and class of Bour type surfaces

For $\mathbb{R}^{2,1}$, the set of roots of a polynomial $Q(x, y, z) = 0$ gives an algebraic surface.
Degree and class of Bour type surfaces

- For $\mathbb{R}^{2,1}$, the set of roots of a polynomial $Q(x, y, z) = 0$ gives an algebraic surface.
- An algebraic surface $f$ is said to be of degree (or order) $n$ when $n = \deg(f)$. 
Degree and class of Bour type surfaces

The tangent plane at a point \((u, v)\) on a surface
\(f(u, v) = (x(u, v), y(u, v), z(u, v))\) is given by

\[
Xx + Yy - Zz + P = 0,
\] (8)

where the Gauss map is \(n = (X(u, v), Y(u, v), Z(u, v))\) and \(P = P(u, v)\).
We have inhomogeneous tangential coordinates $a = X/P$, $b = Y/P$, and $c = Z/P$. 
Degree and class of Bour type surfaces

When we can obtain an implicit equation $\hat{Q}(a, b, c) = 0$ of $f(u, v)$ in tangential coordinates, the maximum degree of the equation gives the class of $f(u, v)$. 
Next, using polynomial elimination methods (in Maple software), we calculate the implicit equations, degrees and classes of spacelike and timelike $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$. 
Degree and class of spacelike Bour of value 2,3,4

From (1), the parametrization of \( \mathcal{B}_2 \) (maximal Enneper surface) is

\[
\mathcal{B}_2(u, v) = \begin{pmatrix}
\frac{1}{3}u^3 - uv^2 + u \\
u^2v - \frac{1}{3}v^3 - v \\
u^2 - v^2
\end{pmatrix} = \begin{pmatrix}
x(u, v) \\
y(u, v) \\
z(u, v)
\end{pmatrix},
\]

where \( u, v \in \mathbb{R} \).
In this section, \( Q_m(x, y, z) = 0 \) denotes the irreducible implicit equation that spacelike or timelike \( \mathcal{B}_m \) will satisfy.
Degree and class of spacelike Bour of value 2,3,4

Then

\[ Q_2(x, y, z) = -64z^9 + 432x^2z^6 - 432y^2z^6 + 1215x^4z^3 \\
+ 6318x^2y^2z^3 - 3888x^2z^5 + 1215y^4z^3 - 3888y^2z^5 \\
+ 1152z^7 + 729x^6 - 2187x^4y^2 - 4374x^4z^2 + 2187x^2y^4 \\
+ 6480x^2z^4 - 729y^6 + 4374y^4z^2 - 6480y^2z^4 \\
- 729x^4z + 1458x^2y^2z + 3888x^2z^3 - 729y^4z \\
+ 3888y^2z^3 - 5184z^5, \]
Degree and class of spacelike Bour of value 2,3,4

- Its degree is \( \text{deg}(B_2) = 9 \).
Degree and class of spacelike Bour of value 2,3,4

- Its degree is \( \deg(\mathcal{B}_2) = 9 \).
- Therefore, \( \mathcal{B}_2 \) is an algebraic maximal surface.
Degree and class of spacelike Bour of value 2,3,4

To find the class of the surface $\mathcal{B}_2$, we obtain

$$P_2(u, v) = \frac{(u^2 + v^2 - 3)(u - v)(u + v)}{3(u^2 + v^2 - 1)},$$

where $P_m(u, v)$ denotes the function as in Equation (8) for spacelike or timelike $\mathcal{B}_m$. 
Degree and class of spacelike Bour of value 2,3,4

To find the class of the surface $\mathcal{B}_2$, we obtain

$$P_2(u, v) = \frac{(u^2 + v^2 - 3)(u - v)(u + v)}{3(u^2 + v^2 - 1)},$$

where $P_m(u, v)$ denotes the function as in Equation (8) for spacelike or timelike $\mathcal{B}_m$.

The inhomogeneous tangential coordinates are

$$a = \frac{6u}{\alpha(u, v)}, \quad b = \frac{6v}{\alpha(u, v)}, \quad c = \frac{6(u^2 + v^2 + 1)}{\alpha(u, v)},$$

where $\alpha(u, v) = (u^2 + v^2 - 3)(u - v)(u + v)$. 
In the tangential coordinates \( a, b, c \),

\[
\hat{Q}_2(a, b, c) = 4a^6 + 9a^4 + 9b^4 + 6a^2b^2c^2 + 12b^2c^3 \\
-3b^4c^2 - 18b^4c - 4a^4b^2 + 18a^4c - 12a^2c^3 \\
-4a^2b^4 - 3a^4c^2 + 18a^2b^2 - 4a^2b^4 + 4b^6,
\]

where \( \hat{Q}_m(a, b, c) = 0 \) denotes the irreducible implicit equation for spacelike or timelike \( \mathcal{B}_m \) in terms of tangential coordinates.
Degree and class of spacelike Bour of value 2,3,4

In the tangential coordinates $a, b, c$,

$$
\hat{Q}_2(a, b, c) = 4a^6 + 9a^4 + 9b^4 + 6a^2b^2c^2 + 12b^2c^3 - 3b^4c^2 - 18b^4c - 4a^4b^2 + 18a^4c - 12a^2c^3 - 4a^2b^4 - 3a^4c^2 + 18a^2b^2 - 4a^2b^4 + 4b^6,
$$

where $\hat{Q}_m(a, b, c) = 0$ denotes the irreducible implicit equation for spacelike or timelike $\mathcal{B}_m$ in terms of tangential coordinates.

Therefore, the class of the spacelike $\mathcal{B}_2$ is $cl(\mathcal{B}_2) = 6$. 
Degree and class of spacelike Bour of value 2,3,4

Similarly,

\[
\mathcal{B}_3(u, v) = \begin{pmatrix}
\frac{u^4}{4} + \frac{v^4}{4} - \frac{3}{2}u^2v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\
\frac{2}{3}u^3v - uv^3 - uv \\
\frac{1}{2}u^3 - 2uv^2
\end{pmatrix} = \begin{pmatrix}
x(u, v) \\
y(u, v) \\
z(u, v)
\end{pmatrix},
\]

\[
\mathcal{B}_4(u, v) = \begin{pmatrix}
\frac{1}{3}u^3 - uv^2 + \frac{1}{5}u^5 - 2u^3v^2 + uv^4 \\
\frac{1}{3}u^3 - 3u^2v^2 + \frac{1}{2}v^4 \\
\frac{1}{2}u^4 - 3u^2v^2 + \frac{1}{2}v^4
\end{pmatrix} = \begin{pmatrix}
x(u, v) \\
y(u, v) \\
z(u, v)
\end{pmatrix},
\]
Degree and class of spacelike Bour of value 2,3,4

and

\[ Q_3(x, y, z) = -43046721z^{16} + 272097792x^3z^{12} \\
-816293376xy^2z^{12} + 3009871872x^6z^8 \\
+14834368512x^4y^2z^8 + (69 \text{ other lower order terms}), \]

\[ Q_4(x, y, z) = -1514571848868138319872z^{25} \\
+92442129475182080000x^4z^{20} \\
-2419276165576171875000000x^4y^{12}z^5 \\
-5546527766851092480000x^2y^2z^{20} \\
-3065257232666015625000000x^{12}y^6z^5 \\
+(233 \text{ other lower order terms}), \]

and their degrees are \( \deg(\mathcal{B}_3) = 16, \deg(\mathcal{B}_4) = 25. \)
Therefore, \( \mathcal{B}_3 \) and \( \mathcal{B}_4 \) are algebraic spacelike maximal surfaces. Furthermore,

\[
P_3(u, v) = \frac{u(u^2 + v^2 - 2)(u^2 - 3v^2)}{(u^2 + v^2 - 1)},
\]

\[
P_4(u, v) = \frac{(3u^2 + 3v^2 - 5)(u^2 - 2uv - v^2)(u^2 + 2uv - v^2)}{30(u^2 + v^2 - 1)},
\]
Degree and class of spacelike Bour of value 2,3,4

and the inhomogeneous tangential coordinates are

\[
\begin{align*}
a &= \frac{12}{\beta(u, v)}, \quad b = \frac{12v}{u\beta(u, v)}, \quad c = \frac{6(u^2 + v^2 + 1)}{u\beta(u, v)} \quad (m = 3), \\
a &= \frac{60u}{\gamma(u, v)}, \quad b = \frac{60v}{\gamma(u, v)}, \quad c = \frac{30(u^2 + v^2 + 1)}{\gamma(u, v)} \quad (m = 4),
\end{align*}
\]

where \( \beta(u, v) = (u^2 + v^2 - 2)(u^2 - 3v^2) \),
\( \gamma(u, v) = (3u^2 + 3v^2 - 5)(u^2 - 2uv - v^2)(u^2 + 2uv - v^2) \).

Then
Degree and class of spacelike Bour of value 2,3,4

\[ \hat{Q}_3(a, b, c) = 9a^8 + 72a^6b^2 - 8a^6c^2 + 144a^4b^4 - 168a^4b^2c^2 - 96a^2b^4c^2 + 96a^2b^2c^4 + 64b^6c^2 - 48b^4c^4 - 72a^7 - 288a^5b^2 + 288a^5c^2 + 288a^3b^2c^2 - 192a^3c^4 + 144a^6, \]
\[ \hat{Q}_4(a, b, c) = -16a^{10} - 8640a^2b^2c^5 - 9000a^4b^4c - 3600a^2b^6c + 12000a^2b^4c^3 + 570a^4b^4c^2 - 180a^2b^6c^2 + 15b^8c^2 - 900b^8 + 1440a^4c^5 + 1440b^4c^5 - 5400a^4b^4 - 3600a^2b^6 + 900b^8c - 2400b^6c^3 - 416a^6b^4 - 416a^4b^6 + 176a^2b^8 - 16b^{10} + 12000a^4b^2c^3 - 3600a^6b^2c - 180a^6b^2c^2 - 3600a^6b^2 + 176a^8b^2 - 2400a^6c^3 + 900a^8c + 15a^8c^2 - 900a^8. \]
Therefore,

\[ cl(\mathcal{B}_3) = 8 \text{ and } cl(\mathcal{B}_4) = 10. \]
Degree and class of timelike Bour of value 2,3,4

From (2), the parametrization of $\mathcal{B}_2$ (timelike Enneper surface) is

$$
\mathcal{B}_2 (u, v) = \begin{pmatrix}
  u^2 + v^2 \\
  u + v - \frac{1}{3} (u^3 + v^3) \\
  -u + v - \frac{1}{3} (u^3 - v^3)
\end{pmatrix} = \begin{pmatrix}
  x(u, v) \\
  y(u, v) \\
  z(u, v)
\end{pmatrix}.
$$

where $u, v \in \mathbb{R}$. Then
Degree and class of timelike Bour of value 2, 3, 4

\[ Q_2(x, y, z) = -16z^9 - 2916y^4z + 4374x^4y^2 - 6318y^2x^2z^3 + 4374x^2y^4 - 15552y^2z^3 - 2916x^4z - 5832x^2y^2z - 20736z^5 + 1152z^7 - 8748x^4z^2 + 8748y^4z^2 + 3888y^2z^5 - 3888x^2z^5 + 15552x^2z^3 + 1215x^4z^3 + 1458x^6 + 216x^2z^6 + 1458y^6 + 1215y^4z^3 + 216y^2z^6 + 12960y^2z^4 + 12960x^2z^4. \]
Degree and class of timelike Bour of value 2, 3, 4

- Its degree is $\deg(\mathcal{B}_2) = 9$. 
Degree and class of timelike Bour of value 2, 3, 4

- Its degree is \( \deg(B_2) = 9 \).
- Hence, \( B_2 \) is an algebraic timelike minimal surface.
To find the class of surface $\mathcal{B}_2$ we obtain

$$P_2(u, v) = \frac{(uv + 3)(u^2 + v^2)}{3(uv + 1)},$$

and the inhomogeneous tangential coordinates are
Degree and class of timelike Bour of value 2,3,4

\[ a = - \frac{(uv - 1)(3uv + 3)}{\hat{\alpha}(u, v)}, \]
\[ b = - \frac{(u + v)(3uv + 3)}{\hat{\alpha}(u, v)}, \]
\[ c = - \frac{(u - v)(3uv + 3)}{\hat{\alpha}(u, v)}, \]

where \( \hat{\alpha}(u, v) = (uv + 1)(uv + 3)(u^2 + v^2) \).
Then

\[ \hat{Q}_2(a, b, c) = 16a^6 + 9a^4 + 36b^4c + 24a^2c^3 
+ 24b^2c^3 - 24a^2b^2c^2 - 12a^4c^2 - 16a^2b^4 - 12b^4c^2 
- 36a^4c + 16a^4b^2 + 9b^4 - 16b^6 - 18a^2b^2. \]

Hence, \( cl(\mathcal{B}_2) = 6 \).
Similarly,

\[ \mathcal{B}_3 (u, v) = \begin{pmatrix} \frac{2}{3} (u^3 + v^3) \\ \frac{1}{2} (u^2 + v^2) - \frac{1}{4} (u^4 + v^4) \\ -\frac{1}{2} (u^2 - v^2) - \frac{1}{4} (u^4 - v^4) \end{pmatrix} = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, \]

\[ \mathcal{B}_4 (u, v) = \begin{pmatrix} \frac{1}{2} (u^4 + v^4) \\ \frac{1}{3} (u^3 + v^3) - \frac{1}{5} (u^5 + v^5) \\ -\frac{1}{3} (u^3 - v^3) - \frac{1}{5} (u^5 - v^5) \end{pmatrix} = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, \]

and
Degree and class of timelike Bour of value 2,3,4

\[ Q_3(x, y, z) = 43046721z^{16} - 1836660096z^{14} + 5435817984x^6z^4 + 602404356096x^4z^8 + 165112971264x^2z^8 + (69 \text{ other lower order terms}), \]

\[ Q_4(x, y, z) = 311836912602146628334544598941564928z^{25} - 3806602937037922709161921373798400000x^4z^{20} - 22839617622227536254971528242790400000x^2y^2z^{20} - 3806602937037922709161921373798400000y^4z^{20} - 2718338279012676739330717777920000000000x^8z^{15} + (233 \text{ other lower order terms}). \]
Degree and class of timelike Bour of value 2,3,4

So

\[ \text{deg}(\mathcal{B}_3) = 16, \quad \text{deg}(\mathcal{B}_4) = 25. \]
Degree and class of timelike Bour of value 2, 3, 4

In the tangential coordinates $a, b, c$,

$$\hat{Q}_3(a, b, c) = 81a^6b^2 - 27a^4b^4 - 72a^4b^2c^2 - 45a^2b^6 - 48a^2b^4c^2 - 9b^8 - 8b^6c^2 - 108a^6b + 180a^4b^3 + 432a^4bc^2 - 36a^2b^5 - 288a^2b^3c^2 - 288a^2bc^4 - 36b^7 - 144b^5c^2 - 96b^3c^4 + 36a^6 - 108a^4b^2 + 108a^2b^4 - 36b^6,$$
Degree and class of timelike Bour of value 2, 3, 4

\[
\hat{Q}_4(a, b, c) = -16a^{10} + 16b^{10} - 450a^8c + 15b^8c^2 \\
-225b^8 - 720a^4c^5 - 1350a^4b^4 + 900a^2b^6 - 450b^8c \\
-1200b^6c^3 - 416a^6b^4 + 416a^4b^6 + 176a^2b^8 \\
-4320a^2b^2c^5 + 4500a^4b^4c - 1800a^2b^6c \\
-6000a^2b^4c^3 + 570a^4b^4c^2 + 180a^2b^6c^2 \\
+6000a^4b^2c^3 - 1800a^6b^2c + 180a^6b^2c^2 \\
-225a^8 - 720b^4c^5 + 900a^6b^2 - 176a^8b^2 \\
+1200a^6c^3 + 15a^8c^2.
\]
Degree and class of timelike Bour of value 2, 3, 4

Therefore,

\[ cl(\mathfrak{B}_3) = 8, \quad cl(\mathfrak{B}_4) = 10. \]
References


References


[8] E. Güler, Bour´s spacelike maximal and timelike minimal surfaces in the three dimensional Lorentz-Minkowski space (presented in GeLoSP 2013, Sao Paulo, Brasil), submitted.


References


References


References


Thank you