$f$-Biminimal Immersions

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Introduction and Preliminaries

Let \((M, g)\) and \((N, h)\) be Riemannian manifolds. A map \(\varphi : (M, g) \rightarrow (N, h)\) is called a harmonic map if it is a critical point of the energy functional

\[
E(\varphi) = \frac{1}{2} \int_M \|d\varphi\|^2 \, d\nu_g.
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Introduction and Preliminaries

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E(\varphi) = \frac{1}{2} \int_M \|d\varphi\|^2 \, d\nu_g.
\]

The map \(\varphi\) is said to be biharmonic if it is a critical point of the bienergy functional

\[
E_2(\varphi) = \frac{1}{2} \int_M \|\tau(\varphi)\|^2 \, d\nu_g,
\]

where \(\tau(\varphi) = tr(\nabla d\varphi)\) is the tension field. If \(\tau(\varphi) = 0\) then \(\varphi\) is called harmonic [Eells-Sampson].
The Euler-Lagrange equation for the bienergy functional were obtained by Jiang in [Jiang-86] by $\tau_2(\varphi) = 0$, where

$$\tau_2(\varphi) = tr(\nabla^N \nabla^N - \nabla^N_{\nabla})\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi),$$

is the bitension field of $\varphi$ and $R^N$ is the curvature tensor of $N$. 
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\[
\tau_2(\varphi) = tr(\nabla^N \nabla^N - \nabla^N_N)\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi), \tag{1}
\]

is the bitension field of \( \varphi \) and \( R^N \) is the curvature tensor of \( N \).

An \textit{f-harmonic map} with a positive function \( f: M \xrightarrow{C^\infty} \mathbb{R} \) is a critical point of \( f \)-energy function

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$$E_f(\varphi) = \frac{1}{2} \int_M f \|d\varphi\|^2 \, d\nu_g.$$ 

Using the Euler-Lagrange equation for the $f$-energy functional, in [OND] and [Course] the $f$-tension field $\tau_f(\varphi)$ was obtained by

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(\text{grad}f). \quad (2)$$
If $\tau_f(\varphi) = 0$ then the map is called $f$-harmonic [Course]. The map $\varphi$ is said to be $f$-biharmonic (see [Lu]) if and only if it is a critical point of the $f$-bienergy functional

$$E_{2,f}(\varphi) = \frac{1}{2} \int_M f \|\tau(\varphi)\|^2 \, d\nu_g.$$
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The Euler-Lagrange equation for the $f$-bienergy functional is given by $\tau_{2,f}(\varphi) = 0$, where $\tau_{2,f}(\varphi)$ is the $f$-bitension field and is defined by

$$\tau_{2,f}(\varphi) = f \tau_2(\varphi) + \Delta f \tau(\varphi) + 2\nabla_{\text{grad} f} \tau(\varphi), \quad (3)$$

(see [Lu]). It can be easily seen that any $f$-harmonic map is $f$-biharmonic. If the map is non-$f$-harmonic $f$-biharmonic then we call it by proper $f$-biharmonic [Lu].
In [Loubeau-Montaldo], Loubeau and Montaldo considered biminimal immersions. They studied biminimal curves in a Riemannian manifold, curves in a space form, and isometric immersions of codimension 1 in a Riemannian manifold. They investigated biminimal surfaces using Riemannian and horizontally homothetic submersions.
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An immersion \( \varphi \), is called \emph{biminimal} (see [Loubeau-Montaldo]) if it is a critical point of the bienergy functional \( E_2(\varphi) \) for variations normal to the image \( \varphi(M) \subset N \), with fixed energy. Equivalently, there exists a constant \( \lambda \in \mathbb{R} \) such that \( \varphi \) is a critical point of the \( \lambda \)-bienergy

\[
E_{2,\lambda}(\varphi) = E_2(\varphi) + \lambda E(\varphi)
\]

for any smooth variation of the map \( \varphi_t : ]-\epsilon, +\epsilon[ \to \varphi_0 = \varphi \), such that \( V = \frac{d\varphi_t}{dt} \big|_{t=0} = 0 \) is normal to \( \varphi(M) \).
The Euler-Lagrange equation for \( \lambda \)-biminimal immersion is,

\[
[\tau_2, \lambda(\varphi)]^\perp = [\tau_2(\varphi)]^\perp - \lambda[\tau(\varphi)]^\perp = 0.
\]  

(5)

for some value of \( \lambda \in \mathbb{R} \), where \([\cdot]^\perp\) denotes the normal component of \([\cdot]\). An immersion is called free biminimal if it is biminimal for \( \lambda = 0 \) [Loubeau-Montaldo].
The Euler-Lagrange equation for $\lambda$-biminimal immersion is,

$$[\tau_{2,\lambda}(\varphi)]^\perp = [\tau_2(\varphi)]^\perp - \lambda[\tau(\varphi)]^\perp = 0.$$  \hspace{1cm} (5)

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In this study, we define $f$-biminimal immersions. We consider $f$-biminimal curves in a Riemannian manifold. We also consider $f$-biminimal submanifolds of codimension 1 in a Riemannian manifold. We give a non-trivial example for an $f$-biminimal Legendre curve in a Sasakian space form and we investigate the Riemannian and horizontally homothetic submersions for proper $f$-biminimal surface in a three dimension Riemannian manifold.
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**Definition 1**

An immersion $\varphi$, is called *$f$-biminimal* if it is a critical point of the $f$-bienergy functional $E_{2,f}(\varphi)$ for variations normal to the image $\varphi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that $\varphi$ is a critical point of the $\lambda$-$f$-bienergy

$$E_{2,\lambda,f}(\varphi) = E_{2,f}(\varphi) + \lambda E_f(\varphi)$$

for any smooth variation of the map $\varphi_t$ which is defined above.
Using the Euler-Lagrange equations for $f$-harmonic and $f$-biharmonic maps, an immersion is $f$-biminimal if

$$[\tau_{2,\lambda,f}(\varphi)]^\perp = [\tau_{2,f}(\varphi)]^\perp - \lambda[\tau_f(\varphi)]^\perp = 0$$

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Using the Euler-Lagrange equations for $f$-harmonic and $f$-biharmonic maps, an immersion is $f$-biminimal if

$$[\tau_2,\lambda,f(\varphi)]^\perp = [\tau_2,f(\varphi)]^\perp - \lambda[\tau_f(\varphi)]^\perp = 0$$  \hspace{1cm} (6)$$

for some value of $\lambda \in \mathbb{R}$.

We call an immersion free $f$-biminimal if it is $f$-biminimal for $\lambda = 0$. If $\varphi$ is a $f$-biminimal but not biminimal immersion then it is called as proper $f$-biminimal.
Let $\gamma : I \subset \mathbb{R} \longrightarrow (M^m, g)$ be a curve parametrized by arc length in a Riemannian manifold $(M^m, g)$. We recall the definition of Frenet frames:

**Definition 2 (Laugwitz)**

The Frenet frame $\{E_i\}_{i=1,2,...,m}$ associated with a curve $\gamma : I \subset \mathbb{R} \longrightarrow (M^m, g)$ is the orthonormalization of the $(m + 1) −$tuple

$$\left\{ \left. \nabla^{(k)} \frac{d\gamma}{dt} \frac{\partial}{\partial t} \right|_{k=0,1,...,m} \right\}$$

described by
\[ E_1 = d\gamma \left( \frac{\partial}{\partial t} \right), \]

\[ \nabla_{\frac{\partial}{\partial t}} E_1 = k_1 E_2, \]

\[ \nabla_{\frac{\partial}{\partial t}} E_i = -k_{i-1} E_{i-1} + k_i E_{i+1}, \quad 2 \leq i \leq m - 1, \]

\[ \nabla_{\frac{\partial}{\partial t}} E_m = -k_{m-1} E_{m-1}, \]

where the function \( \{ k_1 = k > 0, k_2 = \tau, k_3, \ldots, k_{m-1} \} \) are called the curvatures of \( \gamma \). In addition \( E_1 = T = \gamma' \) is the unit tangent vector field to the curve.
\[ E_1 = d\gamma\left(\frac{\partial}{\partial t}\right), \]
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Firstly we have the following proposition for \( f \)-biminimal curve in Riemannian manifold:
Proposition 3

Let $M^m$ be a Riemannian manifold and $\gamma : I \subset \mathbb{R} \rightarrow (M^m, g)$ be an isometric curve. Then $\gamma$ is $f$-biminimal if and only if there exists a real number $\lambda$ such that

$$f \{ (k_1'' - k_1^3 - k_1 k_2^2) - k_1 g(R(E_1, E_2)E_1, E_2) \}$$

$$+ (f'' - \lambda f) k_1 + 2f' k' = 0,$$

(7)

$$f \{ (k_1' k_2 + (k_1 k_2)') - k_1 g(R(E_1, E_2)E_1, E_3) \} + 2f' k_1 k_2 = 0,$$

(8)

$$f \{ k_1 k_2 k_3 - k_1 g(R(E_1, E_2)E_1, E_4) \} = 0,$$

(9)

$$fk_1 g(R(E_1, E_2)E_1, E_j) = 0, \quad 5 \leq j \leq m,$$

(10)

where $R$ is the curvature tensor of $(M^m, g)$. 

Now we investigate $f$-biminimality conditions for a surface or a three dimensional Riemannian manifold with a constant sectional curvature. Then we have the following corollary:
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**Corollary 4**

1) A curve $\gamma$ on a surface of Gaussian curvature $G$ is $f$-biminimal if and only if its signed curvature $k$ satisfies the ordinary differential equation

$$f \left( k'' - k^3 - kG \right) + \left( f'' - \lambda f \right) k + 2f'k' = 0 \quad (11)$$

for some $\lambda \in \mathbb{R}$. 
2) A curve $\gamma$ on Riemannian 3-manifold of constant sectional curvature $c$ is $f$-biminimal if and only if its curvature $k$ and torsion $\tau$ satisfy the system

\begin{align*}
  f \left( k'' - k^3 - k \tau^2 - kc \right) + (f'' - \lambda f) k + 2f'k' &= 0 \\
  f \left( k' \tau + (k \tau)' \right) + 2f'k \tau &= 0.
\end{align*}

(12)

for some $\lambda \in \mathbb{R}$. 
Let $\varphi : M^m \longrightarrow N^{m+1}$ be an isometric immersion. We shall denote by $B$, $\eta$, $A$, $\Delta$ and $H_1 = H\eta$ the second fundamental form, the unit normal vector field, the shape operator, the Laplacian and the mean curvature vector field of $\varphi$ ($H$ the mean curvature function), respectively. Then we have the following proposition:

**Proposition 5**

Let $\varphi : M^m \longrightarrow N^{m+1}$ be an isometric immersion of codimension 1 and $H_1 = H\eta$ its mean curvature vector. Then $\varphi$ is $f$-biminimal if and only if

$$\Delta H - H \|B\|^2 + HRicci(N) + \left(\frac{\Delta f}{f} + 2\text{grad} \ln f - \lambda\right) H = 0.$$
Corollary 6

Let \( \varphi : M^m \rightarrow N^{m+1}(c) \) be an isometric immersion of a Riemannian manifold \( N^{m+1}(c) \) of constant curvature \( c \). Then \( \varphi \) is \( f \)-biminimal if and only if there exists a real number \( \lambda \) such that

\[
\Delta H - \left( m^2 H^2 - s + m(m-2)c - \frac{\Delta f}{f} - 2 \text{grad} \ln f + \lambda \right) H = 0
\]

where \( H \) is the mean curvature and \( s \) the scalar curvature of \( M^m \).

In addition, let \( \varphi : M^2 \rightarrow N^3(c) \) be an isometric immersion from a surface to a three-dimension space form. Then \( \varphi \) is \( f \)-biminimal if and only if

\[
\Delta H - 2H \left( 2H^2 - G - \frac{1}{2} \frac{\Delta f}{f} - \text{grad} \ln f + \frac{1}{2} \lambda \right) = 0
\]
Examples of $f$-Biminimal Surfaces on 3-Dimensional Riemannian Manifolds

Now, we find some examples of $f$-biminimal immersions similar to the methods given in [Loubeau-Montaldo]. A submersion $\varphi : (M, g) \rightarrow (N, h)$ between two Riemannian manifolds in horizontally homothetic if there exists a function $\Lambda : M \rightarrow \mathbb{R}$, the dilation, such that
Examples of $f$-Biminimal Surfaces on 3-Dimensional Riemannian Manifolds

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1) at each point $p \in M$ the differential $d\varphi_p : H_p \longrightarrow T_{\varphi(p)}N$ is a conformal map with factor $\wedge(p)$, i.e.,

$$\wedge^2(p)g(X, Y)(p) = h(d\varphi_p(X), d\varphi_p(Y))(\varphi(p))$$

for all $X, Y, Z \in H_p = \ker_p(d\varphi)^\perp$. 
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2) $X(\wedge^2) = 0$, for all horizontal vector fields [Loubeau-Montaldo].
Lemma 7 (Loubeau-Montaldo)

Let \( \varphi : (M^n, g) \longrightarrow (N^2, h) \) be a horizontally homothetic submersion with \( \wedge \) and minimal fibres and let \( \gamma : I \subset \mathbb{R} \longrightarrow N^2 \) be a curve parametrized by arc length, of signed curvature \( k_\gamma \). Then the codimension-1 submanifold \( S = \varphi^{-1}(\gamma(I)) \subset M \) has mean curvature \( H_s = \frac{\wedge k_\gamma}{n-1} \).
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Using the above lemma, we have the following theorem:
Let \( \varphi : \mathcal{M}^3(c) \to (N^2, h) \) be horizontally homothetic submersion with dilation \( \wedge \), from a space form of constant sectional curvature \( c \) to a surface. Let \( \gamma : I \subset \mathbb{R} \to N^2 \) be a curve parametrized by arc length such that the surface \( S = \varphi^{-1}(\gamma(I)) \subset \mathcal{M}^3 \) has constant Gaussian curvature \( c \). The \( S = \varphi^{-1}(\gamma(I)) \subset \mathcal{M}^3 \) is a \( f \)-biminimal surface (with respect to \( 2c \)) if and only if \( \gamma \) is a free \( f \)-biminimal curve with \( k_\gamma = c_1 e^t \) where \( c_1 \) is a real constant.
Theorem 9

Let $\varphi : M^3(c) \longrightarrow N^2(\overline{c})$ be a Riemannian submersion with minimal fibres from a space of constant sectional curvature $c$ to surface of constant Gaussian curvature $\overline{c}$. Let $\gamma : I \subset \mathbb{R} \longrightarrow N^2$ be a curve parametrized by arc length. Then $S = \varphi^{-1}(\gamma(I)) \subset M^3$ is a $f$-biminimal surface if and only if $\gamma$ is a $f$-biminimal curve with $k_\gamma = c_1 e^t$ where $c_1$ is a real constant.
We consider the Riemannian submersion with totally geodesic fibres, given by the projection onto the first factor 
\( \pi : N^2 \times \mathbb{R} \rightarrow N^2 \) and \( \gamma : I \subset \mathbb{R} \rightarrow N^2 \) be a curve parametrized by arc length. Then we can state the following proposition:
We consider the Riemannian submersion with totally geodesic fibres, given by the projection onto the first factor
\[ \pi : N^2 \times \mathbb{R} \longrightarrow N^2 \] and \[ \gamma : I \subset \mathbb{R} \longrightarrow N^2 \] be a curve parametrized by arc length. Then we can state the following proposition:

**Proposition 10**

The cylinder \( S = \pi^{-1}(\gamma(I)) \) is a proper \( f \)-biminimal surface in \( N^2 \times \mathbb{R} \) if and only if \( \gamma \) is a proper \( f \)-biminimal curve on \( N^2 \) (\( S^2 \) or \( H^2 \)) with curvature \( k = c_1 e^t \), where \( c_1 \) is a real constant.
The three-dimensional Heisenberg space $\mathbb{H}_3$ is the two-step nilpotent Lie group standardly represented in $GL_3(\mathbb{R})$ by

$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ with $x, y, z \in \mathbb{R}$.

It is endowed with the left-invariant metric

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$ (15)
The three-dimensional Heisenberg space $\hat{H}_3$ is the two-step nilpotent Lie group standardly represented in $GL_3(\mathbb{R})$ by

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It is endowed with the left-invariant metric

$$g = dx^2 + dy^2 + (dz - xdy)^2. \quad (15)$$

Let $\pi : \hat{H}_3 \to \mathbb{R}^2$ be the projection $(x, y, z) \to (x, y)$. It is easy to see that $\pi$ is a Riemannian submersion (for more details see [Loubeau-Montaldo]). Take a curve $\gamma(t) = (x(t), y(t))$ in $\mathbb{R}^2$, parametrized by arc length, with signed curvature $k$. 

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Proposition 11

The flat cylinder \( S = \pi^{-1}(\gamma(I)) \subset \hat{H}_3 \) is a proper \( f \)-biminimal surface (with respect to \( \lambda \)) of \( \hat{H}_3 \) if and only if \( \gamma \) is a proper \( f \)-biminimal curve (with respect to \( \lambda + 1 \)) of \( \mathbb{R}^2 \) with curvature \( k = c_1 e^t \), where \( c_1 \) is a real constant.
Let \((M^{2m+1}, \varphi, \xi, \eta, g)\) be a contact metric manifold. If the Nijenhuis tensor of \(\varphi\) equals \(-2d\eta \otimes \xi\), then \((M^{2m+1}, \varphi, \xi, \eta, g)\) is called a Sasakian manifold [Blair]. If a Sasakian manifolds has constant \(\varphi\)–sectional curvature \(c\), then it is called a Sasakian space form. The curvature tensor of a Sasakian space form is given by

\[
R(X, Y)Z = \frac{c + 3}{4} \{g(Y, Z)X - g(X, Z)Y\} \\
+ \frac{c - 1}{4} \{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X\} \\
+ 2g(X, \varphi Y)\varphi Z + \eta(X)\eta(Z)Y \\
- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\]  

(16)
A submanifold of a Sasakian manifold is called an integral submanifold if $\eta(X) = 0$, for every tangent vector $X$. A 1-dimension integral submanifold of a Sasakian manifold is called a Legendre curve of $M$. Hence a curve $\gamma : I \longrightarrow M = (M^{2m+1}, \varphi, \xi, \eta, g)$ is called a Legendre curve if $\eta(T) = 0$, where $T$ is the tangent vector field of $\gamma$ [Blair 2002].
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$\gamma : I \rightarrow M = (M^{2m+1}, \varphi, \xi, \eta, g)$ is called a Legendre curve if $\eta(T) = 0$, where $T$ is the tangent vector field of $\gamma$ [Blair 2002].

**Theorem 12**

Let $\gamma : (a, b) \rightarrow M$ be a non-geodesic Legendre Frenet curve of osculating order $r$ in a Sasakian space form $M = (M^{2m+1}, \varphi, \xi, \eta, g)$. Then $\gamma$ is $f$-biminimal if and only if the following three equations hold.
\[ k''_1 - k^3_1 - k_1 k^2_2 + \frac{(c + 3)}{4} k_1 + 2 k'_1 \frac{f'}{f} + k_1 \frac{f''}{f} \]

\[ -\lambda k_1 + \frac{3(c - 1)}{4} \left[ k_1 g(\varphi T, E_2)^2 \right] \perp = 0, \]

\[ k'_1 k_2 + (k_1 k_2)' + 2 k_1 k_2 \frac{f'}{f} + \frac{3(c - 1)}{4} \left[ k_1 g(\varphi T, E_2) g(\varphi T, E_3) \right] \perp = 0 \]

and

\[ k_1 k_2 k_3 + \frac{3(c - 1)}{4} \left[ k_1 g(\varphi T, E_2) g(\varphi T, E_4) \right] \perp = 0. \]
Let’s recall some notions about the Sasakian space form $\mathbb{R}^{2m+1}(-3)$ [Blair 2002]:

Let us take $M = \mathbb{R}^{2m+1}$ with the standard coordinate functions $(x_1, \ldots, x_m, y_1, \ldots, y_m, z)$, the contact structure $\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y_i dx_i)$, the characteristic vector field $\xi = 2 \frac{\partial}{\partial z}$ and the tensor field $\varphi$ given by

$$\varphi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$
The Riemannian metric is $g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^{m} ((dx_i)^2 + (dy_i)^2)$. Then $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant $\varphi$–sectional curvature $c = -3$ and it is denoted by $\mathbb{R}^{2m+1}(-3)$. The vector fields

$$X_i = 2 \frac{\partial}{\partial y_i}, \quad X_{i+m} = \varphi X_i = 2\left(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}\right), \quad 1 \leq i \leq m, \quad \xi = 2 \frac{\partial}{\partial z},$$

(17)

form a $g$-orthonormal basis and Levi-Civita connection is calculated

$$\nabla_{X_i} X_j = \nabla_{X_{i+m}} X_{j+m} = 0, \quad \nabla_{X_i} X_{j+m} = \delta_{ij} \xi, \quad \nabla_{X_{i+m}} X_j = -\delta_{ij} \xi,$$

$$\nabla_{X_i} \xi = \nabla_{\xi} X_i = -X_{m+i}, \quad \nabla_{X_{i+m}} \xi = \nabla_{\xi} X_{i+m} = X_i$$

(see [Blair]).
Now, let us produce example of proper $f$-biminimal Legendre curves in $\mathbb{R}^5(−3)$:
Now, let us produce example of proper $f$-biminimal Legendre curves in $\mathbb{R}^5(-3)$:

**Example.** Let $\gamma = (\gamma_1, \ldots, \gamma_5)$ be a unit speed Legendre curve in $\mathbb{R}^5(-3)$. The tangent vector field of $\gamma$ is

$$T = \frac{1}{2} \left\{ \gamma_3' X_1 + \gamma_4' X_2 + \gamma_1' X_3 + \gamma_2' X_4 + (\gamma_5' - \gamma_1' \gamma_3 - \gamma_2' \gamma_4) \xi \right\}.$$

Using the above equation, since $\gamma$ is a unit speed Legendre curve we have $\eta(T) = 0$ and $g(T, T) = 1$, that is,

$$\gamma_5' = \gamma_1' \gamma_3 - \gamma_2' \gamma_4$$

and

$$(\gamma_1')^2 + \ldots + (\gamma_5')^2 = 4.$$
For a Legendre curve, we can use the Levi-Civita connection and equation (17) to write

$$\nabla_{T} T = \frac{1}{2} \left( \gamma''' X_1 + \gamma''' X_2 + \gamma'' X_3 + \gamma'' X_4 \right), \quad (18)$$

$$\varphi T = \frac{1}{2} \left( -\gamma' X_1 - \gamma' X_2 + \gamma' X_3 + \gamma' X_4 \right). \quad (19)$$

From equations (18), (19) and $\varphi T \perp E_2$ if and only if

$$\gamma' \gamma''' + \gamma' \gamma''' = \gamma' \gamma'' + \gamma' \gamma''.$$
Finally, we can give the following explicit example:

Let us take $\gamma(t) = (\sin 2t, -\cos 2t, 0, 0, 1)$ in $\mathbb{R}^5(-3)$. Using the above equations and Theorem 12, $\gamma$ is a proper $f$-biminimal Legendre curve with osculating order $r = 2$, $k_1 = 2$, $f = e^t$, $\varphi T \perp E_2$. We can easily check that the conditions of Theorem 12 are verified.
References


Thank you...