Cayley map and Higher Dimensional Representations of Rotations

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Summary of the results

The embeddings of the $\mathfrak{so}(3)$ Lie algebra and the Lie group $\mathrm{SO}(3,\mathbb{R})$ in higher dimensions is an important construction from both mathematical and physical viewpoint. Here we will present a program package for building the generating matrices of the irreducible embeddings of the $\mathfrak{so}(3)$ Lie algebra within $\mathfrak{so}(n)$ for arbitrary dimension $n \geq 3$ relying on the algorithm developed recently by Campoamor-Strursberg [2015]. We will show also that the *Cayley* map applied to $\mathcal{C} \in \mathfrak{so}(n)$ is well defined and generates a subset of $\mathrm{SO}(n)$. Furthermore, we obtain explicit formulas for the images of the *Cayley* map in all cases.

- This research is made within a bigger project which is about parameterizing Lie groups with small dimension and its application in physics.

- Parameterizations are used to describe Lie groups in an easier and more intuitive way. Let G be a finite dimensional Lie group with Lie algebra $\mathfrak{g}$. A vector parameterization of G is a map $\mathfrak{g} \to G$, which is diffeomorphic onto its image. Besides the exponential map, there are other alternatives to achieve parameterization. We make use of the *Cayley* map

$$\mathrm{Cay}(\mathrm{X}) = (\mathcal{I} + \mathrm{X})(\mathcal{I} - \mathrm{X})^{-1}. \tag{1}$$

In Donchev et al, 2015 the *Cayley* maps for the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ and the corresponding Lie groups SU(2) and SO(3, $\mathbb{R}$) are examined.

The vector-parameter of *Gibbs* (or *Fedorov*) is a convenient way to represent proper SO(3, $\mathbb{R}$) rotations. A rotation of angle θ about an axis $\mathbf{n}$ is represented by the vector $\mathbf{c} = \tan\dfrac{\theta}{2}\mathbf{n}$. Any proper SO(3, $\mathbb{R}$) rotation is expressed in the terms of $\mathbf{c}$ in the following manner

$$\mathcal{R}(\mathbf{c}) = \frac{2}{1+c^2}\begin{pmatrix} 1+c_1^2 & c_1c_2-c_3 & c_1c_3+c_2 \\ c_1c_2+c_3 & 1+c_2^2 & c_2c_3-c_1 \\ c_1c_3-c_2 & c_2c_3+c_1 & 1+c_3^2 \end{pmatrix} - \mathcal{I}. \qquad (2)$$

However, one has to be careful when half-turns occur because they are not represented by regular *Gibbs* vectors. We will denote a half-turn about an axis $\mathbf{n}$ by $\mathcal{O}(\mathbf{n})$. The $SO(3,\mathbb{R})$ matrix that corresponds to $\mathcal{O}(\mathbf{n})$ is given by

$$\mathcal{R} = 2\begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{pmatrix} - \mathcal{I}. \tag{3}$$

If $\mathbf{c}$ and $\mathbf{a}$ represent the rotations $\mathcal{R}(\mathbf{c}), \mathcal{R}(\mathbf{a})$, the composition law in vector-parameter form is given by

$$\mathcal{R}(\tilde{\mathbf{c}}) = \mathcal{R}(\mathbf{a})\mathcal{R}(\mathbf{c}), \qquad \widetilde{\mathbf{c}} = \widetilde{\mathbf{c}}(\mathbf{a},\mathbf{c}) = \frac{\mathbf{a}+\mathbf{c}+\mathbf{a}\times\mathbf{c}}{1-\mathbf{a}.\mathbf{c}}\cdot \tag{4}$$

Equation (4) is beautiful, simple and computationally cheap. It takes at most 12 multiplications. In comparison the usual multiplication of two quaternions take 16.

By the means of the *Cayley* maps of the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$, a vector parameter form [Donchev et all, 2015] of *Wigner*'s group homomorphism $W : \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ is derived. After pulling back the group multiplication in SU(2) by the *Cayley* map $\mathrm{Cay}_{\mathfrak{su}(2)} : \mathfrak{su}(2) \to \mathrm{SU}(2)$, explicit formulae for W and for two sections of W are derived. The derived vector-parameterization of SU(2) has the advantage to represent all rotations, including the half-turns. Also the derived composition law is always defined.
An arbitrary $\mathfrak{su}(2)$ element is represented in the following way

$$\mathcal{A} = \mathrm{a}_1 s_1 + \mathrm{a}_2 s_2 + \mathrm{a}_3 s_3 = -\frac{\mathrm{i}}{2}\mathbf{a}\cdot\mathbf{s} \in \mathfrak{su}(2) \tag{5}$$

$s_i = -\frac{\mathrm{i}}{2}\sigma_i, i = 1,2,3$ and $\sigma_i, i = 1,2,3$ can be viewed as Pauli's matrices.

Theorem from Donchev, Mladenova & Mladenov, 2015

Let $\mathcal{U}_1(\mathbf{c}), \mathcal{U}_2(\mathbf{a}) \in \mathrm{SU}(2)$ are the images of $\mathcal{A}_1 = \mathbf{c}\cdot\mathbf{s}$ and $\mathcal{A}_2 = \mathbf{a}\cdot\mathbf{s}$ under the *Cayley* map where $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$. Let

$$\mathcal{U}_3(\langle \mathbf{a}, \mathbf{c}\rangle_{\mathrm{SU}(2)}) = \mathcal{U}_2(\mathbf{a}).\mathcal{U}_1(\mathbf{c}) \tag{6}$$

denote the composition of $\mathcal{U}_2(\mathbf{a})$ and $\mathcal{U}_1(\mathbf{c})$ in SU(2). The corresponding vector-parameter $\tilde{\mathbf{a}} \in \mathbb{R}^3$, for which $\mathrm{Cay}_{\mathfrak{su}(2)}(\mathcal{A}_3) = \mathcal{U}_3$, $\mathcal{A}_3 = \tilde{\mathbf{a}} \cdot \mathbf{s}$ is

$$\tilde{\mathbf{a}} = \frac{\left(1 - \frac{\mathrm{c}^2}{4}\right)\mathbf{a} + \left(1 - \frac{\mathrm{a}^2}{4}\right)\mathbf{c} + 4\frac{\mathbf{a}}{2} \times \frac{\mathbf{c}}{2}}{1 - 2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2} + \frac{\mathrm{a}^2}{4}\frac{\mathrm{c}^2}{4}} . \tag{7}$$

Product of rotations	Result	Condition	Compound rotation
$\mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1)$	$\mathbf{c}_3 = \dfrac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - \mathbf{c}_2.\mathbf{c}_1}$, $[\mathbf{n}_3] = [\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1]$,	$\mathbf{c}_2.\mathbf{c}_1 \neq 1$ $\mathbf{c}_2.\mathbf{c}_1 = 1$	$\mathcal{R}(\mathbf{c}_3)$ $\mathcal{O}(\mathbf{n}_3)$
$\mathcal{R}(\mathbf{c}_2)\mathcal{O}(\mathbf{n}_1)$	$\mathbf{c}_3 = -\dfrac{\mathbf{n}_1 + \mathbf{c}_2 \times \mathbf{n}_1}{\mathbf{c}_2.\mathbf{n}_1}$, $[\mathbf{n}_3] = [\mathbf{n}_1 + \mathbf{c}_2 \times \mathbf{n}_1]$,	$\mathbf{c}_2.\mathbf{n}_1 \neq 0$ $\mathbf{c}_2.\mathbf{n}_1 = 0$	$\mathcal{R}(\mathbf{c}_3)$ $\mathcal{O}(\mathbf{n}_3)$
$\mathcal{O}(\mathbf{n}_2)\mathcal{R}(\mathbf{c}_1)$	$\mathbf{c}_3 = -\dfrac{\mathbf{n}_2 + \mathbf{n}_2 \times \mathbf{c}_1}{\mathbf{n}_2.\mathbf{c}_1}$, $[\mathbf{n}_3] = [\mathbf{n}_2 + \mathbf{n}_2 \times \mathbf{c}_1]$,	$\mathbf{n}_2.\mathbf{c}_1 \neq 0$ $\mathbf{n}_2.\mathbf{c}_1 = 0$	$\mathcal{R}(\mathbf{c}_3)$ $\mathcal{O}(\mathbf{n}_3)$
$\mathcal{O}(\mathbf{n}_2)\mathcal{O}(\mathbf{n}_1)$	$\mathbf{c}_3 = -\dfrac{\mathbf{n}_2 \times \mathbf{n}_1}{\mathbf{n}_2.\mathbf{n}_1}$, $[\mathbf{n}_3] = [\mathbf{n}_2 \times \mathbf{n}_1]$,	$\mathbf{n}_2.\mathbf{n}_1 \neq 0$ $\mathbf{n}_2.\mathbf{n}_1 = 0$	$\mathcal{R}(\mathbf{c}_3)$ $\mathcal{O}(\mathbf{n}_3)$

If $H(\mathbf{c}_1), H(\mathbf{c}_1)$ are two SO(2,1) elements represented by the vector parameters and $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_1.(\eta\mathbf{c}_1) \neq 1, \mathbf{c}_2.(\eta\mathbf{c}_2) \neq 1$ and $1 + \mathbf{c}_2.(\eta\mathbf{c}_1) \neq 0$. Then

$$H(\mathbf{c}_3) = H(\mathbf{c}_2)H(\mathbf{c}_1), \quad \mathbf{c}_3 = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle_{\mathrm{SO}(2,1)} = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_2.(\eta\mathbf{c}_1)} \quad (8)$$

where $\mathbf{c}_2 \curlywedge \mathbf{c}_1 := \eta(\mathbf{c}_2 \times \mathbf{c}_1)$. Equation (8) is the vector-parameter form of SO(2,1) obtained by the parameterization given by the *Cayley* map. The same result was obtained independently by usage of pseudo-quaternions.

Pseudo half-turns are also not covered by this parameterization. Also, the case $\mathbf{c}_2.(\eta\mathbf{c}_1) = -1$ is not covered, which corresponds to the result being a pseudo half-turn. In Donchev et all [2015] the *Cayley* map in the covering group SU$(1,1)$ is used to extend this composition law.

Theorem from Donchev, Mladenova & Mladenov, 2015

Let $M, \mathcal{A} \in \mathfrak{su}(1,1)$

$$M = \mathbf{m}.\mathbf{E}, \quad \mathbf{m} = (m_1, m_2, m_3), \qquad \mathcal{A} = \mathbf{a}.\mathbf{E}, \quad \mathbf{a} = (a_1, a_2, a_3)$$

be such that $\Delta_{\mathbf{m}} \neq 0, \Delta_{\mathbf{a}} \neq 0$ and

$$(\mathbf{a}.(\eta\mathbf{a}))(\mathbf{m}.(\eta\mathbf{m})) + 8\mathbf{a}.(\eta\mathbf{m}) + 16 \neq 0. \tag{9}$$

Let $\mathcal{L}(\mathbf{m}) = \mathrm{Cay}_{\mathfrak{su}(1,1)}(M), \mathcal{W}(\mathbf{a}) = \mathrm{Cay}_{\mathfrak{su}(1,1)}(\mathcal{A})$. Then, if $\tilde{\mathcal{L}} = \mathcal{W}.\mathcal{L}$ is the composition of the images in SU(1,1) then $\tilde{\mathcal{L}} = \mathrm{Cay}_{\mathfrak{su}(1,1)}(\tilde{\mathcal{A}})$ where $\tilde{\mathcal{A}} = \tilde{\mathbf{m}}.\mathbf{E}$ and

$$\tilde{\mathbf{m}} = \frac{(1 + \frac{\mathbf{m}}{2} \cdot (\eta\frac{\mathbf{m}}{2}))\mathbf{a} + (1 + \frac{\mathbf{a}}{2} \cdot (\eta\frac{\mathbf{a}}{2}))\mathbf{m} + \mathbf{a} \curlywedge \mathbf{m}}{1 + 2\frac{\mathbf{a}}{2} \cdot (\eta\frac{\mathbf{m}}{2}) + (\frac{\mathbf{a}}{2} \cdot (\eta\frac{\mathbf{a}}{2}))(\frac{\mathbf{m}}{2} \cdot (\eta\frac{\mathbf{m}}{2}))}. \tag{10}$$

Product of pseudo rotation	Compound rotations	Conditions	Results
$\mathcal{R}_h(\mathbf{c}_2)\mathcal{R}_h(\mathbf{c}_1)$	$\mathcal{R}_h(\mathbf{c})$	$\mathbf{c}_2.\eta\mathbf{c}_1 \neq -1$	$\mathbf{c} = \dfrac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_2.\eta\mathbf{c}_1}$
	$\mathcal{O}_h(\mathbf{m})$	$\mathbf{c}_2.\eta\mathbf{c}_1 = -1$	$\mathbf{m} = -2\dfrac{\eta\mathbf{c}_2 + \eta\mathbf{c}_1 - (\eta\mathbf{c}_2) \curlywedge (\eta\mathbf{c}_1)}{\sqrt{1 - \mathbf{c}_2.\eta\mathbf{c}_2}\sqrt{1 - \mathbf{c}_1.\eta\mathbf{c}_1}}$
$\mathcal{O}_h(\mathbf{m}_2)\mathcal{R}_h(\mathbf{c}_1)$	$\mathcal{R}_h(\mathbf{c})$	$\mathbf{m}_2.\mathbf{c}_1 \neq 0$	$\mathbf{c} = \eta\dfrac{\mathbf{m}_2 - \mathbf{m}_2 \curlywedge (\eta\mathbf{c}_1)}{\mathbf{m}_2.\mathbf{c}_1}$
	$\mathcal{O}_h(\mathbf{m})$	$\mathbf{m}_2.\mathbf{c}_1 = 0$	$\mathbf{m} = -\dfrac{\mathbf{m}_2 - \mathbf{m}_2 \curlywedge (\eta\mathbf{c}_1)}{\sqrt{1 - \mathbf{c}_1.\eta\mathbf{c}_1}}$
$\mathcal{R}_h(\mathbf{c}_2)\mathcal{O}_h(\mathbf{m}_1)$	$\mathcal{R}_h(\mathbf{c})$	$\mathbf{c}_2.\mathbf{m}_1 \neq 0$	$\mathbf{c} = \eta\dfrac{\mathbf{m}_1 - (\eta\mathbf{c}_2) \curlywedge \mathbf{m}_1}{\mathbf{c}_2.\mathbf{m}_1}$
	$\mathcal{O}_h(\mathbf{m})$	$\mathbf{c}_2.\mathbf{m}_1 = 0$	$\mathbf{m} = -\dfrac{\mathbf{m}_1 - (\eta\mathbf{c}_2) \curlywedge \mathbf{m}_1}{\sqrt{1 - \mathbf{c}_2.\eta\mathbf{c}_2}}$
$\mathcal{O}_h(\mathbf{m}_2)\mathcal{O}_h(\mathbf{m}_1)$	$\mathcal{R}_h(\mathbf{c})$	$\mathbf{m}_1 \neq \mathbf{m}_2$	$\mathbf{c} = -\dfrac{\mathbf{m}_2 \times \mathbf{m}_1}{\mathbf{m}_2.\eta\mathbf{m}_1}$
	$\mathcal{I}$	$\mathbf{m}_1 = \mathbf{m}_2$	$\mathbf{c} = \mathbf{0}$

The obtained parameterizations of $\mathrm{SO}(3,\mathbb{R}), \mathrm{SU}(2), \mathrm{SO}(2,1)$ and $\mathrm{SU}(1,1)$ via the *Cayley* map led also to the following additional results:

- One needs at most 12 multiplications and 18 additions to perform the extended composition law. In comparison, the standard quaternion multiplications takes 16 multiplications.
- Explicit form of *Cartan*'s theorem is obtained for $\mathrm{SO}(3,\mathbb{R})$ using the extended vector-parameter form.
- Explicit form of *Cartan*'s theorem is formulated and proved for the hyperbolic $\mathrm{SO}(2,1)$ elements. An arbitrary such element is decomposed into product of two pseudo half-turns.
- The problem for taking a square root in $\mathrm{SO}(2,1)$ is fully and explicitly solved.

Recall the standard $\mathbb{R}$-basis $\mathbf{J}_3 = \{J_{3|1}, J_{3|2}, J_{3|3}\}$ of $\mathfrak{so}(3)$

$$J_{3|1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, J_{3|2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, J_{3|3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Recall that an embedding $j : \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ of Lie algebras is called irreducible [Dynkin, 1952] if the lowest dimensional irreducible representation Γ of $\tilde{\mathfrak{g}}$ remains irreducible when restricted to $\mathfrak{g}$.

Campoamor-Stursberg, 2015 derived explicit formulas for real irreducible representations of the algebra $\mathfrak{so}(3)$ into $\mathfrak{so}(n)$ for $n \geq 3$. To do this, he uses the explicit embedding $\mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{sl}(n, \mathbb{C})$.

Let us denote the constructed in [Campoamor, 2015] embedding by

$$j_n : \mathfrak{so}(3) \hookrightarrow \mathfrak{so}(n) \tag{11}$$

Its nature is different in terms of the different parity of n. Three different cases can be considered:

- $n = 2m + 1$ for $m \in \mathbb{N}$
- $n = 4r + 2$ for $r \in \mathbb{N}$
- $n = 4r + 2$ for $r \in \frac{1}{2}\mathbb{N}$.

Let us denote by $\mathrm{J}_{i|n}, i = 1, 2, 3, \mathbf{J}_n = \{\mathrm{J}_{n|1}, \mathrm{J}_{n|2}, \mathrm{J}_{n|3}\}$ the images of $\mathrm{J}_{3|i}, i = 1, 2, 3$ under the embedding j_n. Let us denote the coefficients

$$a_l^m = \sqrt{\frac{l(2m+1-l)}{4}}, \qquad 0 \le l \le m. \tag{12}$$

Here we present refined formulas for computing $\mathrm{J}_{n|1}, \mathrm{J}_{n|2}, \mathrm{J}_{n|3}$ for all $n \geq 3$.

For $n = 2m+1$, $m \in \mathbb{N}$ we have

$$(\mathrm{J}_{n|1})_{k,l} = \left(\delta^l_{k+1} a^m_{\left[\frac{k}{2}\right]} + \delta^{l+3}_k a^m_{\left[\frac{k-2}{2}\right]}\right)\left(\frac{1+(-1)^k}{2}\right) + \left(\delta^l_n \delta^{n-1}_k - \delta^l_{n-1}\delta^n_k\right) \tag{13}$$

$$\times \left(a^m_m + \sqrt{\frac{m^2+m}{2}}\right) - \left(\delta^l_{k+3} a^m_{\left[\frac{k+1}{2}\right]} + \delta^{l+1}_k a^m_{\left[\frac{k-1}{2}\right]}\right)\left(\frac{1+(-1)^{k-1}}{2}\right)$$

$$(\mathrm{J}_{n|2})_{k,l} = \left(\delta^l_n \delta^{n-2}_k - \delta^l_{n-2}\delta^n_k\right)\left(a^m_m + \sqrt{\frac{m^2+m}{2}}\right) - \left(\delta^l_{k+2} a^m_{\left[\frac{k+1}{2}\right]} + \delta^{l+2}_k a^m_{\left[\frac{k-1}{2}\right]}\right)$$

$$(\mathrm{J}_{n|3})_{k,l} = \frac{(1+(-1)^k)\delta^{l+1}_k (n+1-k) - (1+(-1)^{k-1})\delta^{k+1}_l (n-k)}{4}$$

where $1 \le k, l \le n$ and $[x]$ denotes the integer part of x.

For $n = 4r + 2,\ r = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ and $1 \le k, l \le n$

$$\begin{aligned}
(\mathrm{J}_{n|1})_{k,l} &= (\delta^{l}_{k+3} a^{r}_{[\frac{k+1}{2}]} + \delta^{l+1}_{k} a^{r}_{[\frac{k-1}{2}]}) \left(\frac{1+(-1)^{k-1}}{2}\right) \\
&\quad - \left(\frac{1+(-1)^{k}}{2}\right) (\delta^{l}_{k+1} a^{r}_{[\frac{k}{2}]} + \delta^{l+3}_{k} a^{r}_{[\frac{k-2}{2}]}) \\
(\mathrm{J}_{n|2})_{k,l} &= \delta^{l}_{k+2} a^{r}_{[\frac{k+1}{2}]} + \delta^{l+2}_{k} a^{r}_{[\frac{k-1}{2}]} \\
(\mathrm{J}_{n|3})_{k,l} &= \frac{(1+(-1)^{k})\delta^{l+1}_{k}(n+1-k) - (1+(-1)^{k-1})\delta^{k+1}_{l}(n-k)}{4}.
\end{aligned} \tag{14}$$

Besides the correction of the technical errors we changed the signs of $\mathbf{J}_{n|1}$ and $\mathbf{J}_{n|2}$ (this is an automorphism of $\mathfrak{so}(3)$) in order to ensure consistency with the case $n = 3$.

$$\mathbf{c}.\mathbf{J}_4 = \frac{1}{2}\begin{pmatrix} 0 & -c_3 & -c_2 & -c_1 \\ c_3 & 0 & c_1 & -c_2 \\ c_2 & -c_1 & 0 & c_3 \\ c_1 & c_2 & -c_3 & 0 \end{pmatrix}. \tag{15}$$

$$\mathbf{c}.\mathbf{J}_5 = \begin{pmatrix} 0 & -2c_3 & c_2 & c_1 & 0 \\ 2c_3 & 0 & -c_1 & c_2 & 0 \\ -c_2 & c_1 & 0 & -c_3 & -\sqrt{3}c_2 \\ -c_1 & -c_2 & c_3 & 0 & \sqrt{3}c_1 \\ 0 & 0 & \sqrt{3}c_2 & -\sqrt{3}c_1 & 0 \end{pmatrix}. \tag{16}$$

$$\mathbf{c}.\mathbf{J}_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\sqrt{2}c_3 & -c_2 & -c_1 & 0 & 0 \\ \sqrt{2}c_3 & 0 & c_1 & -c_2 & 0 & 0 \\ c_2 & -c_1 & 0 & 0 & -c_2 & -c_1 \\ c_1 & c_2 & 0 & 0 & c_1 & -c_2 \\ 0 & 0 & c_2 & -c_1 & 0 & \sqrt{2}c_3 \\ 0 & 0 & c_1 & c_2 & -\sqrt{2}c_3 & 0 \end{pmatrix} \tag{17}$$

We will consider the *Cayley* defined on Im j_n, i.e.,

$$\mathrm{Cay}(\mathcal{C}) = (\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} \tag{18}$$

for arbitrary Im $j_n \ni \mathcal{C} = \mathbf{c}.\mathbf{J}_n = c_1.\mathrm{J}_{n|1} + c_2.\mathrm{J}_{n|2} + c_3.\mathrm{J}_{n|3}$, where

$$\mathbf{c} = (c_1, c_2, c_3), \qquad \mathbf{c}^2 = c_1^2 + c_2^2 + c_3^2 = |\mathbf{c}|^2 = c^2. \tag{19}$$

We will derive explicit formulas for (18) in the different cases for the parity of $n \geq 3$.

Let $n = 2m+1$, $m \geq 1$. The characteristic polynomial of an arbitrary matrix $\mathcal{C} = \mathbf{c}.\mathbf{J}_n$ is [Fedorov, Campoamor]

$$
\begin{aligned}
-p_{2m+1}(\lambda) &= \lambda(\lambda^2 + 1^2c^2)\ldots(\lambda^2 + m^2c^2) = \lambda\prod_{t=1}^{m}(\lambda^2 + t^2c^2) \\
&= \lambda^{2m+1} + \alpha_{2m-1}c^2\lambda^{2m-1} + \ldots + \alpha_1 c^{2m}\lambda \qquad (20) \\
&= \lambda^{2m+1} + \sum_{t=1}^{m}\alpha_{2m+1-2t}c^{2t}\lambda^{2m+1-2t}
\end{aligned}
$$

where $\alpha_1, \alpha_3, \ldots \alpha_{2m-1}$ are the coefficients of the polynomial p_{2m+1}. One can derive formulas for them using *Vieta*'s formulas for the polynomial

$$
g(\mu) = \mu^m + \alpha_{2m-1}\mu^{m-1} + \alpha_{2m-3}\mu^{m-2} + \ldots + \alpha_3\mu + \alpha_1 \qquad (21)
$$

obtained by $\dfrac{-p_{2m+1}(\lambda)}{\lambda c^{2m}}$ after a substitution of $\dfrac{\lambda^2}{c^2}$ for μ. This is the polynomial of degree m with simple roots $-1^2, -2^2, \ldots - m^2$, i.e., $g(\mu) = (\mu + 1^2)\ldots(\mu + m^2)$.

We have that

$$\alpha_{2m+1-2t} = \sum_{1 \le i_1 < \ldots < i_t \le m} i_1^2 \ldots i_t^2, \qquad t = 1, 2, \ldots, m. \qquad (22)$$

For example, the closed forms of α_1, $\alpha_{2m-3}, \alpha_{2m-1}$ for $m \geq 2$ are $\alpha_1 = (m!)^2$,

$$\alpha_{2m-1} = \frac{m(m+1)(2m+1)}{6}, \quad \alpha_{2m-3} = \frac{m(m^2-1)(4m^2-1)(5m+6)}{180}.$$

More explicit expressions and relations for the coefficients $\alpha_{2m+1-2t}$, $t = 1, \ldots, m$ can be sought via the usage of *Bernouli* coefficients and the generalized harmonic coefficients $H_{m,2} = \sum\limits_{s=1}^{m} \dfrac{1}{m^2}$. For example, $\alpha_3 = (m!)^2 H_{m,2}$.

Theorem 1

For an arbitrary $n = 2m + 1, m \geq 1$ the Cayley map (18) is well-defined on $\operatorname{Im} j_n$ *and the following explicit formula holds true*

$$\operatorname{Cay}(\mathcal{C}) = \mathcal{I} + 2\sum_{s=0}^{m-1} \frac{1 + \sum\limits_{k=1}^{m-s-1} \alpha_{2k+1} c^{2m-2k}}{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}} (\mathcal{C}^{2s+1} + \mathcal{C}^{2s+2}). \tag{23}$$

for all $\mathcal{C} = \mathbf{c}.\mathbf{J}_n \in \operatorname{Im} j_n$. *Also, the map* Cay *takes values in* $\mathrm{SO}(n)$.

Example: $n = 5$

In the special case $n = 5$ the characteristic polynomial of the matix $\mathcal{C}_5 = \mathbf{c}.\mathbf{J}_5$ form (16) is

$$p_5(\lambda) = -\lambda^5 - 4c^2\lambda^3 - 5c^4\lambda$$

and the explicit formula for the *Cayley* map reads as

$$\mathrm{Cay}_{\mathfrak{so}(5)\restriction\mathfrak{so}(3)}(\mathcal{C}) = \mathcal{I} + 2\frac{5c^2+1}{4c^4+5c^2+1}(\mathcal{C}+\mathcal{C}^2) + 2\frac{1}{4c^4+5c^2+1}(\mathcal{C}^3+\mathcal{C}^4). \tag{24}$$

Proof of Theorem 3, I

We need to prove that $\mathcal{I} - \mathcal{C}$ is invertable and to find an explicit formula for it. We will seek a formula for $(\mathcal{I} - \mathcal{C})^{-1}$ via the *ansatz*

$$(\mathcal{I} - \mathcal{C})^{-1} = x_0\mathcal{I} + x_1\mathcal{C} + \ldots x_{2m}\mathcal{C}^{2m}. \qquad (25)$$

We seek such numbers $x_0, \ldots x_{2m}$ that $\mathcal{I} = (\mathcal{I} - \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1}$. Taking into account (20) we calculate

$$\begin{aligned}
\mathcal{I} &= (\mathcal{I} - \mathcal{C})(x_0\mathcal{I} + x_1\mathcal{C} + \ldots + x_{2m}\mathcal{C}^{2m}) \\
&= x_0\mathcal{I} + (x_1 - x_0)\mathcal{C} + (x_2 - x_1)\mathcal{C}^2 + \ldots + (x_{2m} - x_{2m-1})\mathcal{C}^{2m} - x_{2m}\mathcal{C}^{2m+1} \\
&= x_0\mathcal{I} + (x_1 - x_0 + x_{2m}\alpha_1 c^{2m})\mathcal{C} + (x_2 - x_1)\mathcal{C}^2 + \ldots \qquad (26) \\
&\quad + (x_{2m-1} - x_{2m-2} + x_{2m}\alpha_{2m-1}c^2)\mathcal{C}^{2m-1} + (x_{2m} - x_{2m-1})\mathcal{C}^{2m} \\
&= x_0\mathcal{I} + \sum_{s=0}^{m-1}(x_{2s+1} - x_{2s} + x_{2m}\alpha_{2s+1}c^{2m-2s})\mathcal{C}^{2s+1} + \sum_{s=0}^{m-1}(x_{2s+2} - x_{2s+1})\mathcal{C}^{2s+2}.
\end{aligned}$$

Proof of Theorem 3, II

From (26) we directly obtain a linear system of equations for the unknown $x_0, \ldots, x_{2m}$ consisting of $2m + 1$ equations which can be split into the following two parts:

$$\begin{array}{rcl} x_2 & = & x_1 \\ x_4 & = & x_3 \\ & \cdots & \\ x_{2m} & = & x_{2m-1} \end{array} \quad \text{and} \quad \begin{array}{rcrcl} x_0 & & & = & 1 \\ x_1 & - & x_0 & = & -x_{2m}\alpha_1 c^{2m} \\ x_3 & - & x_2 & = & -x_{2m}\alpha_3 c^{2m-2} \\ & \cdots & & & \\ x_{2m-1} & - & x_{2m-2} & = & -x_{2m}\alpha_{2m-1} c^2. \end{array} \tag{27}$$

Proof of Theorem 3, III

Step by step we obtain $x_0 = 1$

$$\begin{aligned} x_2 &= x_1 &&= 1 - x_{2m}\alpha_1 c^{2m} \\ x_4 &= x_3 &&= 1 - x_{2m}(\alpha_1 c^{2m} + \alpha_3 c^{2m-2}) \\ \ldots \\ x_{2m} &= x_{2m-1} &&= 1 - x_{2m}(\alpha_1 c^{2m} + \alpha_3 c^{2m-2} + \ldots + \alpha_{2m-1} c^2) \end{aligned} \quad (28)$$

Summing up all of equations in (27), we obtain

$$\begin{aligned} x_{2m} = x_{2m-1} &= 1 - x_{2m}(\alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}) \\ &= 1 + x_{2m}(p_{2m+1}(1) + 1) \end{aligned}$$

and thus

$$x_{2m} = -\frac{1}{p_{2m+1}(1)} = \frac{1}{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}}.$$

Proof of Theorem 3, IV

Note that

$$-p_{2m+1}(1) = p_{2m+1}(-1) = (1 + c^2)(1 + 4c^2)\dots(1 + m^2c^2) > 0$$

for all $\mathbf{c} \in \mathbb{R}^3$. Substituting this result in (28) gives

$$\begin{aligned}
x_2 &= x_1 = \frac{1 + \alpha_{2m-1}c^2 + \dots + \alpha_3 c^{2m-2}}{1 + \alpha_{2m-1}c^2 + \dots + \alpha_1 c^{2m}} \\
x_4 &= x_3 = \frac{1 + \alpha_{2m-1}c^2 + \dots + \alpha_5 c^{2m-4}}{1 + \alpha_{2m-1}c^2 + \dots + \alpha_1 c^{2m}} \\
&\dots \\
x_{2m} &= x_{2m-1} = \frac{1}{1 + \alpha_{2m-1}c^2 + \dots + \alpha_1 c^{2m}}
\end{aligned} \tag{29}$$

We just obtained that for all $\mathbf{c} \in \mathbb{R}^3$ $(\mathcal{I} - \mathcal{C})^{-1}$ exists and

$$(\mathcal{I} - \mathcal{C})^{-1} = \mathcal{I} + \sum_{s=0}^{m-1} \frac{1 + \sum_{k=1}^{m-s-1} \alpha_{2k+1} c^{2m-2k}}{1 + \alpha_{2m-1}c^2 + \dots + \alpha_1 c^{2m}} (\mathcal{C}^{2s+1} + \mathcal{C}^{2s+2}). \tag{30}$$

Proof of Theorem 3, V

Now it is a straightforward, but tedious calculation of to calculate $(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}$ as a polynomial of $\mathcal{C}$, which leads to the formula (23). It is curious that formulae for $(\mathcal{I}-\mathcal{C})^{-1}$ and Cay$(\mathcal{C})$ are so alike. We are left to prove that $(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}$ is an SO(n) matrix. Using the fact that $\mathcal{C}^t = -\mathcal{C}$ and the fact that the matrices $\mathcal{I}-\mathcal{C}$ and $\mathcal{I}+\mathcal{C}$ commute. we obtain

$$((\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1})^t(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1} = ((\mathcal{I}-\mathcal{C})^{-1})^t(\mathcal{I}+\mathcal{C})^t(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}$$
$$= (\mathcal{I}+\mathcal{C})^{-1}(\mathcal{I}-\mathcal{C})(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1} = (\mathcal{I}+\mathcal{C})^{-1}(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}$$
$$= \mathcal{I}.$$

Furthermore

$$\det(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1} = \frac{\det(\mathcal{I}+\mathcal{C})}{\det(\mathcal{I}-\mathcal{C})} = \frac{\det(\mathcal{I}+\mathcal{C})}{\det(\mathcal{I}+\mathcal{C})^t} = 1.$$

The proof is complete.

The case of even dimension

Let $n = 4r + 2$, $r \in \mathbb{N}$. The characteristic polynomial of an arbitrary matrix $\mathcal{C} = \mathbf{c}.\mathbf{J}_n$ is

$$\begin{aligned} p_{4r+2}(\lambda) &= \lambda^2(\lambda^2 + 1^2c^2)^2(\lambda^2 + 2^2c^2)^2 \dots (\lambda^2 + r^2c^2)^2 = \lambda^2 \prod_{t=1}^{r}(\lambda^2 + t^2c^2)^2 \\ &= \lambda^{4r+2} + \beta_{4r}c^2\lambda^{4r} + \dots + \beta_2 c^{4r}\lambda^2 \qquad (31) \\ &= \lambda^{4r+2} + \sum_{t=1}^{2r} \beta_{4r+2-2t}c^{2t}\lambda^{4r+2-2t} \end{aligned}$$

where $1, \beta_2, \beta_4, \dots \beta_{4r}$ are the coefficients of the polynomial p_{4r+2}. One can derive formulas for them using *Vieta*'s formulas for the polynomial

$$h(\nu) = \nu^{2r} + \beta_{4r}\mu^{2\nu-1} + \dots \beta_4\nu + \beta_2 \qquad (32)$$

obtained by $\dfrac{p_{4r+2}(\lambda)}{\lambda^2}$ after a substitution of $\dfrac{\lambda^2}{c^2}$ for $\mu^2 = \nu$. The distinct roots of h are $-1^2, -2^2, \dots, -r^2$ and all of them are with a multiplicity of two.

Theorem 2

For an arbitrary $n = 4r + 2, r \in \mathbb{N}$ the Cayley map (18) is well-defined on $\operatorname{Im} j_n$ and the following explicit formula holds true:

$$\operatorname{Cay}(\mathcal{C}) = \mathcal{I} + 2\mathcal{C} + 2\sum_{s=1}^{r} \frac{1 + \sum\limits_{k=1}^{2r-2s-1} \beta_{2k+2} c^{4r-2k}}{1 + \beta_{4r} c^2 + \ldots + \beta_2 c^{4r}} (\mathcal{C}^{2s} + \mathcal{C}^{2s+1}). \tag{33}$$

for all $\mathcal{C} = \mathbf{c}.\mathbf{J}_n \in \operatorname{Im} j_n$. Also, the map Cay takes values in $\operatorname{SO}(n)$.

Example: $n = 6$

In the special case $n = 6$ the characteristic polynomial of the matix $\mathcal{C}_6 = \mathbf{c}.\mathbf{J}_6$ form (16) is

$$p_6(\lambda) = \lambda^6 + 2c^2\lambda^4 + c^4\lambda^2$$

and the explicit formula for the *Cayley* map reads as

$$\mathrm{Cay}_{\mathfrak{so}(6)\restriction\mathfrak{so}(3)}(\mathcal{C}) = \mathcal{I} + 2\mathcal{C} + \frac{2c^2+1}{1+2c^2+c^4}(\mathcal{C}^2+\mathcal{C}^3) + 2\frac{1}{1+2c^2+c^4}(\mathcal{C}^4+\mathcal{C}^5). \tag{34}$$

Let $r_1 = \dfrac{2k-1}{2}, k \geq 1$ be a half-integer. Then $n = 4\dfrac{2k-1}{2} + 2 = 4r$ for $r \in \mathbb{N}$. In these series we will obtain all representations in dimensions n that are multiple of 4. The characteristic polynomial of an arbitrary matrix $\mathcal{C} = \mathbf{c}.\mathbf{J}_n$ is

$$p_{4r}(\lambda) = \prod_{t=1}^{r}(\lambda^2 + (\frac{2t-1}{2})^2 c^2)^2 \tag{35}$$

$$= \lambda^{4r} + \gamma_{4r-2}c^2\lambda^{4r-2} + \ldots + \gamma_0 c^{4r}\lambda^0 = \lambda^{4r+2} + \sum_{t=1}^{2r} \gamma_{4r-2t}c^{2t}\lambda^{4r-2t}.$$

Expressions for the coefficients $1, \gamma_{4r-2}, \gamma_{4r-4}, \ldots \gamma_2$ of the polynomial p_{4r+2} can be obtained using *Vieta*'s formulas for the polynomial

$$u(\nu) = \nu^{2r} + \gamma_{4r-2}\mu^{2\nu-1} + \ldots \gamma_4\nu^1 + \gamma_2 \tag{36}$$

obtained by $\dfrac{p_{4r}(\lambda)}{c^{4r}}$ after a substitution of $\dfrac{\lambda^2}{c^2}$ for $\mu^2 = \nu$. The distinct roots of u are $-(\frac{1}{2})^2, -(\frac{3}{2})^2, \ldots, -(\frac{r}{2})^2$ and all of them are with a multiplicity of two.

Theorem 3

For an arbitrary $n = 4r + 2, r \in \mathbb{N}$ the Cayley map (18) is well-defined on $\operatorname{Im} j_n$ *and the following explicit formula holds true:*

$$\operatorname{Cay}(\mathcal{C}) = -\mathcal{I} + 2\sum_{s=0}^{r} \frac{1 + \sum\limits_{k=1}^{2r-2s-1} \gamma_{2k} c^{4r-2k}}{1 + \gamma_{4r-2} c^2 + \ldots + \gamma_0 c^{4r}} (\mathcal{C}^{2s} + \mathcal{C}^{2s+1}). \quad (37)$$

for all $\mathcal{C} = \mathbf{c}.\mathbf{J}_n \in \operatorname{Im} j_n$. *Also, the map* Cay *takes values in* $\mathrm{SO}(n)$.

The special case $n = 4$

The *Hamilton–Cayley* theorem for $\mathcal{C}$ reads as

$$\mathcal{C}^4 + \frac{c^2}{2}\mathcal{C}^2 + \frac{c^4}{16}\mathcal{I} = \mathcal{O} \Rightarrow \mathcal{C}^4 = -\frac{c^2}{2}\mathcal{C}^2 - \frac{c^4}{16}\mathcal{I}. \qquad (38)$$

Despite this fact one directly can check that in this special case ($n = 4$) we have also the stronger equality $\mathcal{C}^2 = \dfrac{c^2}{4}\mathcal{I}$. Using this, let us find an explicit expression for the *Cayley* map Cay as a polynomial of degree 1 instead of 3 as expected from Theorem 37. We have that $(\mathcal{I} - \mathcal{C})^{-1} = \dfrac{4}{4 + c^2}(\mathcal{I} + \mathcal{C})$, which leads to

$$\mathrm{Cay}(\mathcal{C}) = (\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} = \frac{4 - c^2}{4 + c^2}\mathcal{I} + \frac{8}{4 + c^2}\mathcal{C}. \qquad (39)$$

Obviously, the *Cayley* map is defined for all $c \in \mathbb{R}^3$.

How do we can extract the vector $\mathbf{c}$ from a given matrix $\mathcal{R}_4(\mathbf{c}) = \mathrm{Cay}(\mathcal{C})$? We have that

$$\operatorname{tr}\mathcal{R}_4(\mathbf{c}) = 3\frac{4-c^2}{4+c^2} \Rightarrow \frac{1}{4+c^2} = \frac{3-\operatorname{tr}\mathcal{R}_4(\mathbf{c})}{24} \tag{40}$$

and thus if we consider $\mathcal{A} = \mathcal{R}_4(\mathbf{c}) - \mathcal{R}_4^t(\mathbf{c}) = \frac{16}{4+c^2}\mathcal{C}$ than we have

$$2\mathcal{C}(\mathbf{c}) = \frac{3}{3-\operatorname{tr}\mathcal{R}_4(\mathbf{c})}\mathcal{A} \tag{41}$$

and $\mathbf{c} = -2(\mathcal{C}_{1,4}, \mathcal{C}_{1,3}, \mathcal{C}_{1,2})$.

Let $\mathcal{C} = \mathbf{c}.\mathbf{J}_4$ and $\mathcal{A} = \mathbf{a}.\mathbf{J}_4$ be two arbitrary elements of Im j_4. Let $\mathcal{R}_\mathbf{c}$ and $\mathcal{R}_\mathbf{a}$ be the images of these matrices under the *Cayley* map, i.e.,

$$\mathcal{R}_\mathbf{c} = \mathrm{Cay}(\mathcal{C}), \qquad \mathcal{R}_\mathbf{a} = \mathrm{Cay}(\mathcal{A}).$$

Let $\mathcal{R} = \mathcal{R}_\mathbf{a}\mathcal{R}_\mathbf{c}$ be their composition in SO(4). We want to find an element $\widetilde{\mathcal{C}} = \widetilde{\mathbf{c}}.\mathbf{J}_4$ such that $\mathrm{Cay}(\widetilde{\mathcal{C}}) = \mathcal{R} = \mathcal{R}_{\widetilde{\mathbf{c}}}$.
Let us note that the direct calculation gives

$$\mathcal{A}.\mathcal{C} = -\frac{\mathbf{a}.\mathbf{c}}{4}\mathcal{I} + \frac{\mathbf{a} \times \mathbf{c}}{2} \cdot \mathbf{J}_4. \tag{42}$$

This leads to very similar calculations as in the case of SU(2) vector parameter. They lead to the following composition of $\mathrm{SO}(3,\mathbb{R})$ matrices in SO(4)

$$\widetilde{\mathbf{c}} = \langle \mathbf{a}, \mathbf{c} \rangle_{\mathrm{Cay}_{\Im j_4}} = \frac{\left(1 - \frac{\mathrm{c}^2}{4}\right)\mathbf{a} + \left(1 - \frac{\mathrm{a}^2}{4}\right)\mathbf{c} + 4\frac{\mathbf{a}}{2} \times \frac{\mathbf{c}}{2}}{1 - 2\frac{\mathbf{a}}{2} \cdot \frac{\mathbf{c}}{2} + \frac{\mathrm{a}^2}{4}\frac{\mathrm{c}^2}{4}} \cdot \tag{43}$$

Further development

- We need to check if the $\mathrm{SO}(3, \mathbb{R})$ half-turns are realized as $\mathrm{SO}(n)$ matrices for $n \geq 4$ via the *Cayley* map applied for the embedded $\mathfrak{so}(3)$ algebra into $\mathfrak{so}(n)$.
- We will investigate what is the composition law for $n > 4$
- We will investigate if there are efficient formulas to extract the matrix $\mathcal{C}$ generating the three-dimensional rotation matrix $\mathcal{R}_n(\mathbf{c}) \in \mathrm{SO}(n)$
- We will investigate if some important operators' representations in dimension $n > 3$ are more convenient.

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