Cayley map and Higher Dimensional Representations of Rotations

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Introduction and Prerequisites

The embedding $\mathfrak{so}(3) \hookrightarrow \mathfrak{so}(n)$ and examples
The Cayley map and higher order representations of rotations

The presentation

Summary of the results

The embeddings of the $\mathfrak{so}(3)$ Lie algebra and the Lie group $\text{SO}(3, \mathbb{R})$ in higher dimensions is an important construction from both mathematical and physical viewpoint. Here we will present a program package for building the generating matrices of the irreducible embeddings of the $\mathfrak{so}(3)$ Lie algebra within $\mathfrak{so}(n)$ for arbitrary dimension $n \geq 3$ relying on the algorithm developed recently by Campoamor-Strursberg [2015]. We will show also that the Cayley map applied to $C \in \mathfrak{so}(n)$ is well defined and generates a subset of $\text{SO}(n)$. Furthermore, we obtain explicit formulas for the images of the Cayley map in all cases.
This research is made within a bigger project which is about parameterizing Lie groups with small dimension and its application in physics.

Parameterizations are used to describe Lie groups in an easier and more intuitive way. Let \( G \) be a finite dimensional Lie group with Lie algebra \( \mathfrak{g} \). A vector parameterization of \( G \) is a map \( \mathfrak{g} \rightarrow G \), which is diffeomorphic onto its image. Besides the exponential map, there are other alternatives to achieve parameterization. We make use of the Cayley map

\[
\text{Cay}(X) = (I + X)(I - X)^{-1}.
\]
In Donchev et al, 2015 the Cayley maps for the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ and the corresponding Lie groups SU(2) and SO(3, $\mathbb{R}$) are examined.

The vector-parameter of Gibbs (or Fedorov) is a convenient way to represent proper SO(3, $\mathbb{R}$) rotations. A rotation of angle $\theta$ about an axis $\mathbf{n}$ is represented by the vector $\mathbf{c} = \tan \frac{\theta}{2} \mathbf{n}$. Any proper SO(3, $\mathbb{R}$) rotation is expressed in the terms of $\mathbf{c}$ in the following manner

$$
\mathcal{R}(\mathbf{c}) = \frac{2}{1 + c^2} \begin{pmatrix}
1 + c_1^2 & c_1 c_2 - c_3 & c_1 c_3 + c_2 \\
c_1 c_2 + c_3 & 1 + c_2^2 & c_2 c_3 - c_1 \\
c_1 c_3 - c_2 & c_2 c_3 + c_1 & 1 + c_3^2
\end{pmatrix} - \mathcal{I}.
$$

(2)
However, one has to be careful when half-turns occur because they are not represented by regular Gibbs vectors. We will denote a half-turn about an axis \( \mathbf{n} \) by \( \mathcal{O}(\mathbf{n}) \). The \( \text{SO}(3, \mathbb{R}) \) matrix that corresponds to \( \mathcal{O}(\mathbf{n}) \) is given by

\[
\mathcal{R} = 2 \begin{pmatrix}
  \mathbf{n}_1^2 & \mathbf{n}_1 \mathbf{n}_2 & \mathbf{n}_1 \mathbf{n}_3 \\
  \mathbf{n}_1 \mathbf{n}_2 & \mathbf{n}_2^2 & \mathbf{n}_2 \mathbf{n}_3 \\
  \mathbf{n}_1 \mathbf{n}_3 & \mathbf{n}_2 \mathbf{n}_3 & \mathbf{n}_3^2
\end{pmatrix} - \mathcal{I}.
\]

(3)

If \( \mathbf{c} \) and \( \mathbf{a} \) represent the rotations \( \mathcal{R}(\mathbf{c}), \mathcal{R}(\mathbf{a}) \), the composition law in vector-parameter form is given by

\[
\mathcal{R}(\tilde{\mathbf{c}}) = \mathcal{R}(\mathbf{a})\mathcal{R}(\mathbf{c}), \quad \tilde{\mathbf{c}} = \tilde{\mathbf{c}}(\mathbf{a}, \mathbf{c}) = \frac{\mathbf{a} + \mathbf{c} + \mathbf{a} \times \mathbf{c}}{1 - \mathbf{a} \cdot \mathbf{c}}.
\]

(4)

Equation (4) is beautiful, simple and computationally cheap. It takes at most 12 multiplications. In comparison the usual multiplication of two quaternions take 16.
By the means of the Cayley maps of the Lie algebras \( \mathfrak{su}(2) \) and \( \mathfrak{so}(3) \), a vector parameter form [Donchev et al, 2015] of Wigner’s group homomorphism \( W : \text{SU}(2) \to \text{SO}(3, \mathbb{R}) \) is derived. After pulling back the group multiplication in \( \text{SU}(2) \) by the Cayley map \( \text{Cay}_{\mathfrak{su}(2)} : \mathfrak{su}(2) \to \text{SU}(2) \), explicit formulae for \( W \) and for two sections of \( W \) are derived. The derived vector-parameterization of \( \text{SU}(2) \) has the advantage to represent all rotations, including the half-turns. Also the derived composition law is always defined. An arbitrary \( \mathfrak{su}(2) \) element is represented in the following way

\[
\mathcal{A} = a_1 s_1 + a_2 s_2 + a_3 s_3 = -\frac{i}{2} \mathbf{a} \cdot \mathbf{s} \in \mathfrak{su}(2)
\]  

\( s_i = -\frac{i}{2} \sigma_i \), \( i = 1, 2, 3 \) and \( \sigma_i \), \( i = 1, 2, 3 \) can be viewed as Pauli’s matrices.
Theorem from Donchev, Mladenova & Mladenov, 2015

Let $\mathcal{U}_1(c), \mathcal{U}_2(a) \in \text{SU}(2)$ are the images of $A_1 = c \cdot s$ and $A_2 = a \cdot s$ under the Cayley map where $a, c \in \mathbb{R}^3$. Let

$$\mathcal{U}_3(\langle a, c \rangle_{\text{SU}(2)}) = \mathcal{U}_2(a) \cdot \mathcal{U}_1(c)$$

(6)

denote the composition of $\mathcal{U}_2(a)$ and $\mathcal{U}_1(c)$ in $\text{SU}(2)$. The corresponding vector-parameter $\tilde{a} \in \mathbb{R}^3$, for which $\text{Cay}_{\text{su}(2)}(A_3) = \mathcal{U}_3$, $A_3 = \tilde{a} \cdot s$ is

$$\tilde{a} = \left(1 - \frac{c^2}{4}\right)a + \left(1 - \frac{a^2}{4}\right)c + 4\frac{a}{2} \times \frac{c}{2}$$

(7)

$$\frac{1 - 2 \frac{a}{2} \cdot \frac{c}{2} + \frac{a^2 c^2}{4}}{4}$$
### Introduction and Prerequisites
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### Vector-parameter forms of $SO(3, \mathbb{R})$ and $SU(2)$
Vector-parameter forms of $SO(2,1)$ and $SU(1,1)$

More benefits of the Cayley map

<table>
<thead>
<tr>
<th>Product of rotations</th>
<th>Result</th>
<th>Condition</th>
<th>Compound rotation</th>
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<tbody>
<tr>
<td>$\mathcal{R}(c_2)\mathcal{R}(c_1)$</td>
<td>$c_3 = \frac{c_2 + c_1 + c_2 \times c_1}{1 - c_2 \cdot c_1}$, $[n_3] = [c_2 + c_1 + c_2 \times c_1]$,</td>
<td>$c_2 \cdot c_1 \neq 1$</td>
<td>$\mathcal{R}(c_3)$</td>
</tr>
<tr>
<td>$\mathcal{R}(c_2)\mathcal{O}(n_1)$</td>
<td>$c_3 = -\frac{n_1 + c_2 \times n_1}{c_2 \cdot n_1}$, $[n_3] = [n_1 + c_2 \times n_1]$,</td>
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Cayley map and Higher Dimensional Representations of Rotations
If $H(c_1), H(c_1)$ are two SO(2,1) elements represented by the vector parameters and $c_1, c_2$ and $c_1.(\eta c_1) \neq 1, c_2.(\eta c_2) \neq 1$ and $1 + c_2.(\eta c_1) \neq 0$. Then

$$H(c_3) = H(c_2)H(c_1), \quad c_3 = \langle c_2, c_1 \rangle_{SO(2,1)} = \frac{c_2 + c_1 + c_2 \lhd c_1}{1 + c_2.(\eta c_1)} \quad (8)$$

where $c_2 \lhd c_1 := \eta(c_2 \times c_1)$. Equation (8) is the vector-parameter form of SO(2,1) obtained by the parameterization given by the Cayley map. The same result was obtained independently by usage of pseudo-quaternions.

Pseudo half-turns are also not covered by this parameterization. Also, the case $c_2.(\eta c_1) = -1$ is not covered, which corresponds to the result being a pseudo half-turn. In Donchev et al [2015] the Cayley map in the covering group SU(1, 1) is used to extend this composition law.
Theorem from Donchev, Mladenova & Mladenov, 2015

Let $M, A \in \mathfrak{su}(1, 1)$

$M = m \cdot E, \quad m = (m_1, m_2, m_3), \quad A = a \cdot E, \quad a = (a_1, a_2, a_3)$

be such that $\Delta_m \neq 0, \Delta_a \neq 0$ and

$$(a \cdot (\eta a))(m \cdot (\eta m)) + 8a \cdot (\eta m) + 16 \neq 0. \quad (9)$$

Let $\mathcal{L}(m) = \text{Cay}_{\mathfrak{su}(1,1)}(M), \mathcal{W}(a) = \text{Cay}_{\mathfrak{su}(1,1)}(A)$. Then, if

$\tilde{\mathcal{L}} = \mathcal{W} \cdot \mathcal{L}$ is the composition of the images in SU(1, 1) then

$\tilde{\mathcal{L}} = \text{Cay}_{\mathfrak{su}(1,1)}(\tilde{A})$ where $\tilde{A} = \tilde{m} \cdot E$ and

$$\tilde{m} = \frac{(1 + \frac{m}{2} \cdot (\eta \frac{m}{2}))a + \frac{1 + \frac{a}{2} \cdot (\eta \frac{a}{2})}{1 + 2\frac{a}{2} \cdot (\eta \frac{a}{2}) + \frac{a}{2} \cdot (\eta \frac{a}{2})}(\frac{m}{2} \cdot (\eta \frac{m}{2}))}{m + a \cdot m} \quad (10)$$
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<td>$R_h(c)$</td>
<td>$c_2 \cdot \eta c_1 \neq -1$</td>
<td>$c = \frac{c_2 + c_1 + c_2 \wedge c_1}{1 + c_2 \cdot \eta c_1}$</td>
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<tr>
<td></td>
<td>$O_h(m)$</td>
<td>$c_2 \cdot \eta c_1 = -1$</td>
<td>$m = -2 \frac{\eta c_2 + \eta c_1 - (\eta c_2) \wedge (\eta c_1)}{\sqrt{1 - c_2 \cdot \eta c_2} \sqrt{1 - c_1 \cdot \eta c_1}}$</td>
</tr>
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Cayley map and Higher Dimensional Representations of Rotations
The obtained parameterizations of $\text{SO}(3, \mathbb{R})$, $\text{SU}(2)$, $\text{SO}(2,1)$ and $\text{SU}(1,1)$ via the Cayley map led also to the following additional results:

- One needs at most 12 multiplications and 18 additions to perform the extended composition law. In comparison, the standard quaternion multiplications takes 16 multiplications.
- Explicit form of Cartan’s theorem is obtained for $\text{SO}(3, \mathbb{R})$ using the extended vector-parameter form.
- Explicit form of Cartan’s theorem is formulated and proved for the hyperbolic $\text{SO}(2,1)$ elements. An arbitrary such element is decomposed into product of two pseudo half-turns.
- The problem for taking a square root in $\text{SO}(2,1)$ is fully and explicitly solved.
Recall the standard $\mathbb{R}$-basis $J_3 = \{J_3|1, J_3|2, J_3|3\}$ of $\mathfrak{so}(3)$

$J_3|1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, J_3|2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, J_3|3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

Recall that an embedding $j : \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ of Lie algebras is called irreducible [Dynkin, 1952] if the lowest dimensional irreducible representation $\Gamma$ of $\tilde{\mathfrak{g}}$ remains irreducible when restricted to $\mathfrak{g}$.

Campoamor-Stursberg, 2015 derived explicit formulas for real irreducible representations of the algebra $\mathfrak{so}(3)$ into $\mathfrak{so}(n)$ for $n \geq 3$. To do this, he uses the explicit embedding $\mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{sl}(n, \mathbb{C})$. 
Let us denote the constructed in [Campoamor, 2015] embedding by
\[ j_n : so(3) \hookrightarrow so(n) \] (11)

Its nature is different in terms of the different parity of \( n \). Three different cases can be considered:

- \( n = 2m + 1 \) for \( m \in \mathbb{N} \)
- \( n = 4r + 2 \) for \( r \in \mathbb{N} \)
- \( n = 4r + 2 \) for \( r \in \frac{1}{2} \mathbb{N} \).

Let us denote by \( J_{i|n}, i = 1, 2, 3, J_n = \{J_{n|1}, J_{n|2}, J_{n|3}\} \) the images of \( J_{3|i}, i = 1, 2, 3 \) under the embedding \( j_n \). Let us denote the coefficients
\[ a_i^m = \sqrt{\frac{l(2m + 1 - l)}{4}}, \quad 0 \leq l \leq m. \] (12)

Here we present refined formulas for computing \( J_{n|1}, J_{n|2}, J_{n|3} \) for all \( n \geq 3 \).
For $n = 2m + 1$, $m \in \mathbb{N}$ we have

\[
(J_{n|1})_{k,l} = (\delta_{k+1}^l a_{\frac{k}{2}}^m + \delta_{k}^{l+3} a_{\frac{k-2}{2}}^m) \left(\frac{1 + (-1)^k}{2}\right) + (\delta_{n}^l \delta_{k}^{n-1} - \delta_{n-1}^l \delta_{k}^{n})
\]

(13)

\[
\times \left( a_{m}^m + \sqrt{\frac{m^2 + m}{2}} \right) - (\delta_{k+3}^l a_{\frac{k+1}{2}}^m + \delta_{k}^{l+1} a_{\frac{k-1}{2}}^m) \left(\frac{1 + (-1)^{k-1}}{2}\right)
\]

\[
(J_{n|2})_{k,l} = (\delta_{n}^l \delta_{k}^{n-2} - \delta_{n-2}^l \delta_{k}^{n}) \left( a_{m}^m + \sqrt{\frac{m^2 + m}{2}} \right) - (\delta_{k+2}^l a_{\frac{k+1}{2}}^m + \delta_{k}^{l+2} a_{\frac{k-1}{2}}^m)
\]

\[
(J_{n|3})_{k,l} = \frac{(1 + (-1)^k)\delta_{k}^{l+1}(n + 1 - k) - (1 + (-1)^{k-1})\delta_{k}^{l+1}(n - k)}{4}
\]

where $1 \leq k, l \leq n$ and $[x]$ denotes the integer part of $x$. 
For $n = 4r + 2$, $r = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ and $1 \leq k, l \leq n$

\[
(J_{n|1})_{k,l} = (\delta^l_{k+3} a^r_{\left[\frac{k+1}{2}\right]} + \delta^l_{k+1} a^r_{\left[\frac{k-1}{2}\right]}) \left(\frac{1 + (-1)^{k-1}}{2}\right) \\
- \left(\frac{1 + (-1)^{k}}{2}\right) (\delta^l_{k+1} a^r_{\left[\frac{k}{2}\right]} + \delta^l_{k} a^r_{\left[\frac{k-2}{2}\right]})
\]

\[
(J_{n|2})_{k,l} = \delta^l_{k+2} a^r_{\left[\frac{k+1}{2}\right]} + \delta^l_{k} a^r_{\left[\frac{k-1}{2}\right]}
\]

\[
(J_{n|3})_{k,l} = \frac{(1 + (-1)^{k})\delta^l_{k+1}(n + 1 - k) - (1 + (-1)^{k-1})\delta^l_{k}(n - k)}{4}.
\]

Besides the correction of the technical errors we changed the signs of $J_{n|1}$ and $J_{n|2}$ (this is an automorphism of $so(3)$) in order to ensure consistency with the case $n = 3$. 
The embedding $\mathfrak{so}(3) \rightarrow \mathfrak{so}(n)$ and examples

The **Cayley map** and higher order representations of rotations

### Explicit formulas

#### Examples

\[ \mathbf{c} \cdot \mathbf{J}_4 = \frac{1}{2} \begin{pmatrix} 0 & -c_3 & -c_2 & -c_1 \\ c_3 & 0 & c_1 & -c_2 \\ c_2 & -c_1 & 0 & c_3 \\ c_1 & c_2 & -c_3 & 0 \end{pmatrix}. \tag{15} \]

\[ \mathbf{c} \cdot \mathbf{J}_5 = \begin{pmatrix} 0 & -2c_3 & c_2 & c_1 & 0 \\ 2c_3 & 0 & -c_1 & c_2 & 0 \\ -c_2 & c_1 & 0 & -c_3 & -\sqrt{3}c_2 \\ -c_1 & -c_2 & c_3 & 0 & \sqrt{3}c_1 \\ 0 & 0 & \sqrt{3}c_2 & -\sqrt{3}c_1 & 0 \end{pmatrix}. \tag{16} \]

\[ \mathbf{c} \cdot \mathbf{J}_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\sqrt{2}c_3 & -c_2 & -c_1 & 0 & 0 \\ \sqrt{2}c_3 & 0 & c_1 & -c_2 & 0 & 0 \\ c_2 & -c_1 & 0 & 0 & -c_2 & -c_1 \\ c_1 & c_2 & 0 & 0 & c_1 & -c_2 \\ 0 & 0 & c_2 & -c_1 & 0 & \sqrt{2}c_3 \\ 0 & 0 & c_1 & c_2 & -\sqrt{2}c_3 & 0 \end{pmatrix}. \tag{17} \]
We will consider the \textit{Cayley} defined on Im \( j_n \), i.e.,

\[
Cay(C) = (I + C)(I - C)^{-1}
\]

for arbitrary Im \( j_n \ni C = c.J_n = c_1.J_n|_1 + c_2.J_n|_2 + c_3.J_n|_3 \), where

\[
C = (c_1, c_2, c_3), \quad C^2 = c_1^2 + c_2^2 + c_3^2 = |C|^2 = c^2.
\]

We will derive explicit formulas for (18) in the different cases for the parity of \( n \geq 3 \).
Let \( n = 2m+1, m \geq 1 \). The characteristic polynomial of an arbitrary matrix \( C = c.J_n \) is [Fedorov, Campoamor]

\[
-p_{2m+1}(\lambda) = \lambda(\lambda^2 + 1^2c^2) \ldots (\lambda^2 + m^2c^2) = \lambda \prod_{t=1}^{m}(\lambda^2 + t^2c^2)
\]

\[
= \lambda^{2m+1} + \alpha_{2m-1}c^2\lambda^{2m-1} + \ldots + \alpha_1c^{2m}\lambda
\]

\[
= \lambda^{2m+1} + \sum_{t=1}^{m} \alpha_{2m+1-2t}c^{2t}\lambda^{2m+1-2t}
\]

(20)

where \( \alpha_1, \alpha_3, \ldots \alpha_{2m-1} \) are the coefficients of the polynomial \( p_{2m+1} \). One can derive formulas for them using Vieta’s formulas for the polynomial

\[
g(\mu) = \mu^m + \alpha_{2m-1}\mu^{m-1} + \alpha_{2m-3}\mu^{m-2} + \ldots + \alpha_3\mu + \alpha_1
\]

(21)

obtained by \( \frac{-p_{2m+1}(\lambda)}{\lambda c^{2m}} \) after a substitution of \( \frac{\lambda^2}{c^2} \) for \( \mu \). This is the polynomial of degree \( m \) with simple roots \(-1^2, -2^2, \ldots - m^2\), i.e., \( g(\mu) = (\mu + 1^2) \ldots (\mu + m^2) \).
We have that

\[ \alpha_{2m+1-2t} = \sum_{1 \leq i_1 < \ldots < i_t \leq m} i_1^2 \ldots i_t^2, \quad t = 1, 2, \ldots, m. \quad (22) \]

For example, the closed forms of \( \alpha_1, \alpha_{2m-3}, \alpha_{2m-1} \) for \( m \geq 2 \) are

\[ \alpha_1 = (m!)^2, \]

\[ \alpha_{2m-1} = \frac{m(m + 1)(2m + 1)}{6}, \quad \alpha_{2m-3} = \frac{m(m^2 - 1)(4m^2 - 1)(5m + 6)}{180}. \]

More explicit expressions and relations for the coefficients \( \alpha_{2m+1-2t}, t = 1, \ldots, m \) can be sought via the usage of Bernoulli coefficients and the generalized harmonic coefficients \( H_{m,2} = \sum_{s=1}^{m} \frac{1}{m^2} \). For example,

\[ \alpha_3 = (m!)^2 H_{m,2}. \]
Theorem 1

For an arbitrary \( n = 2m + 1, \ m \geq 1 \) the Cayley map (18) is well-defined on \( \text{Im} \ j_n \) and the following explicit formula holds true

\[
\text{Cay}(C) = I + 2 \sum_{s=0}^{m-1} \frac{1 + \sum_{k=1}^{m-s-1} \alpha_{2k+1} C^{2m-2k}}{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}} (C^{2s+1} + C^{2s+2}).
\]

(23)

for all \( C = c.J_n \in \text{Im} \ j_n \). Also, the map \( \text{Cay} \) takes values in \( \text{SO}(n) \).
Example: $n = 5$

In the special case $n = 5$ the characteristic polynomial of the matrix $C_5 = cJ_5$ form (16) is

$$p_5(\lambda) = -\lambda^5 - 4c^2\lambda^3 - 5c^4\lambda$$

and the explicit formula for the Cayley map reads as

$$\text{Cay}_{so(5)}|_{so(3)}(C) = \mathcal{I} + 2 \frac{5c^2 + 1}{4c^4 + 5c^2 + 1}(C + C^2) + 2 \frac{1}{4c^4 + 5c^2 + 1}(C^3 + C^4).$$

(24)
Proof of Theorem 3, 1

We need to prove that $\mathcal{I} - C$ is invertible and to find an explicit formula for it. We will seek a formula for $(\mathcal{I} - C)^{-1}$ via the ansatz

$$(\mathcal{I} - C)^{-1} = x_0 \mathcal{I} + x_1 C + \ldots x_{2m} C^{2m}.$$  \hspace{1cm} (25)

We seek such numbers $x_0, \ldots x_{2m}$ that $\mathcal{I} = (\mathcal{I} - C)(\mathcal{I} - C)^{-1}$. Taking into account (20) we calculate

$$\mathcal{I} = (\mathcal{I} - C)(x_0 \mathcal{I} + x_1 C + \ldots + x_{2m} C^{2m})$$

$$= x_0 \mathcal{I} + (x_1 - x_0)C + (x_2 - x_1)C^2 + \ldots + (x_{2m} - x_{2m-1})C^{2m} - x_{2m} C^{2m+1}$$

$$= x_0 \mathcal{I} + (x_1 - x_0 + x_{2m} \alpha_1 C^{2m})C + (x_2 - x_1)C^2 + \ldots$$

$$+ (x_{2m-1} - x_{2m-2} + x_{2m} \alpha_{2m-1} C^2)C^{2m-1} + (x_{2m} - x_{2m-1})C^{2m}$$

$$= x_0 \mathcal{I} + \sum_{s=0}^{m-1} (x_{2s+1} - x_{2s} + x_{2m} \alpha_{2s+1} C^{2m-2s})C^{2s+1} + \sum_{s=0}^{m-1} (x_{2s+2} - x_{2s+1})C^{2s+2}. \hspace{1cm} (26)$$
Proof of Theorem 3, II

From (26) we directly obtain a linear system of equations for the unknown $x_0, \ldots, x_{2m}$ consisting of $2m + 1$ equations which can be split into the following two parts:

\[
\begin{align*}
    x_2 &= x_1 \\
    x_4 &= x_3 \\
    &\vdots \\
    x_{2m} &= x_{2m-1}
\end{align*}
\quad \text{and} \quad
\begin{align*}
    x_0 &= 1 \\
    x_1 - x_0 &= -x_{2m} \alpha_1 c^{2m} \\
    x_3 - x_2 &= -x_{2m} \alpha_3 c^{2m-2} \\
    &\vdots \\
    x_{2m-1} - x_{2m-2} &= -x_{2m} \alpha_{2m-1} c^2.
\end{align*}
\]

(27)
Proof of Theorem 3, III

Step by step we obtain $x_0 = 1$

\[
\begin{align*}
    x_2 &= x_1 = 1 - x_{2m} \alpha_1 c^{2m} \\
    x_4 &= x_3 = 1 - x_{2m}(\alpha_1 c^{2m} + \alpha_3 c^{2m-2}) \\
    \vdots \\
    x_{2m} &= x_{2m-1} = 1 - x_{2m}(\alpha_1 c^{2m} + \alpha_3 c^{2m-2} + \ldots + \alpha_{2m-1} c^2) \\
\end{align*}
\]

(28)

Summing up all of equations in (27), we obtain

\[
\begin{align*}
    x_{2m} &= x_{2m-1} = 1 - x_{2m}(\alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}) \\
    &= 1 + x_{2m}(p_{2m+1}(1) + 1) \\
\end{align*}
\]

and thus

\[
    x_{2m} = -\frac{1}{p_{2m+1}(1)} = \frac{1}{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}}.
\]
Proof of Theorem 3, IV

Note that

$$-p_{2m+1}(1) = p_{2m+1}(-1) = (1 + c^2)(1 + 4c^2) \ldots (1 + m^2 c^2) > 0$$

for all \( c \in \mathbb{R}^3 \). Substituting this result in (28) gives

\[
\begin{align*}
  x_2 &= x_1 &= \frac{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_3 c^{2m-2}}{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}} \\
  x_4 &= x_3 &= \frac{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_5 c^{2m-4}}{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}} \\
  \ldots \\
  x_{2m} &= x_{2m-1} &= \frac{1}{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}}
\end{align*}
\]

We just obtained that for all \( c \in \mathbb{R}^3 \) \((I - C)^{-1}\) exists and

\[
(I - C)^{-1} = I + \sum_{s=0}^{m-1} \frac{1 + \sum_{k=1}^{m-s-1} \alpha_{2k+1} c^{2m-2k}}{1 + \alpha_{2m-1} c^2 + \ldots + \alpha_1 c^{2m}} (C^{2s+1} + C^{2s+2}).
\]
Proof of Theorem 3, V

Now it is a straightforward, but tedious calculation of to calculate \((\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1}\) as a polynomial of \(\mathcal{C}\), which leads to the formula (23). It is curious that formulae for 
\(\mathcal{I} - \mathcal{C})^{-1}\) and \(\text{Cay}(\mathcal{C})\) are so alike. We are left to prove that \((\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1}\) is an \(\text{SO}(n)\) matrix. Using the fact that \(\mathcal{C}^t = -\mathcal{C}\) and the fact that the matrices \(\mathcal{I} - \mathcal{C}\) and \(\mathcal{I} + \mathcal{C}\) commute. we obtain

\[
((\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1})^t(\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} = ((\mathcal{I} - \mathcal{C})^{-1})^t(\mathcal{I} + \mathcal{C})^t(\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} \\
= (\mathcal{I} + \mathcal{C})^{-1}(\mathcal{I} - \mathcal{C})(\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} = (\mathcal{I} + \mathcal{C})^{-1}(\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} \\
= \mathcal{I}.
\]

Furthermore

\[
det \, (\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} = \frac{\det (\mathcal{I} + \mathcal{C})}{\det (\mathcal{I} - \mathcal{C})} = \frac{\det (\mathcal{I} + \mathcal{C})}{\det (\mathcal{I} + \mathcal{C})^t} = 1.
\]

The proof is complete.
The case of even dimension

Let \( n = 4r + 2, r \in \mathbb{N} \). The characteristic polynomial of an arbitrary matrix \( C = c \cdot J_n \) is

\[
p_{4r+2}(\lambda) = \lambda^2(\lambda^2 + 1^2 c^2)^2(\lambda^2 + 2^2 c^2)^2 \ldots (\lambda^2 + r^2 c^2)^2 = \lambda^2 \prod_{t=1}^{r}(\lambda^2 + t^2 c^2)^2
\]

\[
= \lambda^{4r+2} + \beta_{4r} c^2 \lambda^{4r} + \ldots + \beta_2 c^{4r} \lambda^2
\]

(31)

\[
= \lambda^{4r+2} + \sum_{t=1}^{2r} \beta_{4r+2-2t} c^{2t} \lambda^{4r+2-2t}
\]

where \( 1, \beta_2, \beta_4, \ldots \beta_{4r} \) are the coefficients of the polynomial \( p_{4r+2} \). One can derive formulas for them using Vieta’s formulas for the polynomial

\[
h(\nu) = \nu^{2r} + \beta_{4r} \mu^{2\nu-1} + \ldots \beta_4 \nu + \beta_2
\]

(32)

obtained by \( \frac{p_{4r+2}(\lambda)}{\lambda^2} \) after a substitution of \( \frac{\lambda^2}{c^2} \) for \( \mu^2 = \nu \). The distinct roots of \( h \) are \(-1^2, -2^2, \ldots, -r^2\) and all of them are with a multiplicity of two.
Theorem 2

For an arbitrary $n = 4r + 2$, $r \in \mathbb{N}$ the Cayley map (18) is well-defined on $\text{Im} \ j_n$ and the following explicit formula holds true:

$$
\text{Cay}(C) = I + 2C + 2 \sum_{s=1}^{r} \frac{2r-2s-1}{1 + \beta_{4r}c^2 + \ldots + \beta_{2r}c^{4r}}(C^{2s} + C^{2s+1}).
$$

(33)

for all $C = c \cdot J_n \in \text{Im} \ j_n$. Also, the map Cay takes values in $\text{SO}(n)$. 
Example: $n = 6$

In the special case $n = 6$ the characteristic polynomial of the matrix $C_6 = c J_6$ form (16) is

$$p_6(\lambda) = \lambda^6 + 2c^2\lambda^4 + c^4\lambda^2$$

and the explicit formula for the Cayley map reads as

$$\text{Cay}_\mathfrak{so}(6)\rvert_{\mathfrak{so}(3)}(C) = I + 2C + \frac{2c^2 + 1}{1 + 2c^2 + c^4}(C^2 + C^3) + 2\frac{1}{1 + 2c^2 + c^4}(C^4 + C^5).$$

(34)
Let \( r_1 = \frac{2k - 1}{2} \), \( k \geq 1 \) be a half-integer. Then \( n = 4 \frac{2k - 1}{2} + 2 = 4r \) for \( r \in \mathbb{N} \). In these series we will obtain all representations in dimensions \( n \) that are multiple of 4. The characteristic polynomial of an arbitrary matrix \( C = cJ_n \) is

\[
p_{4r}(\lambda) = \prod_{t=1}^{r}(\lambda^2 + \left(\frac{2t - 1}{2}\right)^2c^2)^2
\]

\[
= \lambda^{4r} + \gamma_{4r-2}c^2\lambda^{4r-2} + \ldots + \gamma_0 c^{4r}\lambda^0 = \lambda^{4r+2} + \sum_{t=1}^{2r} \gamma_{4r-2t}c^{2t}\lambda^{4r-2t}.
\]

Expressions for the coefficients \( 1, \gamma_{4r-2}, \gamma_{4r-4}, \ldots \gamma_2 \) of the polynomial \( p_{4r+2} \) can be obtained using Vieta’s formulas for the polynomial

\[
u(\nu) = \nu^{2r} + \gamma_{4r-2}\mu^{2\nu-1} + \ldots \gamma_4\nu^1 + \gamma_2
\]

obtained by \( \frac{p_{4r}(\lambda)}{c^{4r}} \) after a substitution of \( \frac{\lambda^2}{c^2} \) for \( \mu^2 = \nu \). The distinct roots of \( u \) are \(-\left(\frac{1}{2}\right)^2, -\left(\frac{3}{2}\right)^2, \ldots, -\left(\frac{r}{2}\right)^2\) and all of them are with a multiplicity of two.
Theorem 3

For an arbitrary \( n = 4r + 2, r \in \mathbb{N} \) the Cayley map (18) is well-defined on \( \text{Im} \, j_n \) and the following explicit formula holds true:

\[
\text{Cay}(C) = -I + 2 \sum_{s=0}^{r} \frac{1 + \sum_{k=1}^{2r-2s-1} \gamma_{2k} c^{4r-2k}}{1 + \gamma_{4r-2} c^2 + \ldots + \gamma_0 c^{4r}} (C^{2s} + C^{2s+1}).
\]

for all \( C = c.J_n \in \text{Im} \, j_n \). Also, the map Cay takes values in \( \text{SO}(n) \).
The special case $n = 4$

The *Hamilton–Cayley* theorem for $C$ reads as

$$C^4 + \frac{c^2}{2}C^2 + \frac{c^4}{16}I = O \Rightarrow C^4 = -\frac{c^2}{2}C^2 - \frac{c^4}{16}I.$$  \hfill (38)

Despite this fact one directly can check that in this special case ($n = 4$) we have also the stronger equality $C^2 = \frac{c^2}{4}I$. Using this, let us find an explicit expression for the *Cayley* map $Cay$ as a polynomial of degree 1 instead of 3 as expected from Theorem 37. We have

$$Cay(C) = (I + C)(I - C)^{-1} = \frac{4 - c^2}{4 + c^2}I + \frac{8}{4 + c^2}C.$$  \hfill (39)

Obviously, the *Cayley* map is defined for all $c \in \mathbb{R}^3$. 

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Cayley map and Higher Dimensional Representations of Rotations
How do we can extract the vector $\mathbf{c}$ from a given matrix $\mathcal{R}_4(\mathbf{c}) = \text{Cay}(\mathcal{C})$? We have that

$$\text{tr} \: \mathcal{R}_4(\mathbf{c}) = 3 \frac{4 - c^2}{4 + c^2} \Rightarrow \frac{1}{4 + c^2} = \frac{3 - \text{tr} \: \mathcal{R}_4(\mathbf{c})}{24} \quad (40)$$

and thus if we consider $\mathcal{A} = \mathcal{R}_4(\mathbf{c}) - \mathcal{R}_4^t(\mathbf{c}) = \frac{16}{4 + c^2} \mathcal{C}$ than we have

$$2\mathcal{C}(\mathbf{c}) = \frac{3}{3 - \text{tr} \: \mathcal{R}_4(\mathbf{c})} \mathcal{A} \quad (41)$$

and $\mathbf{c} = -2(C_{1,4}, C_{1,3}, C_{1,2})$. 
Let $C = c \cdot J_4$ and $A = a \cdot J_4$ be two arbitrary elements of $\text{Im} \, j_4$. Let $R_c$ and $R_a$ be the images of these matrices under the Cayley map, i.e.,

$$R_c = \text{Cay}(C), \quad R_a = \text{Cay}(A).$$

Let $R = R_a R_c$ be their composition in $\text{SO}(4)$. We want to find an element $\tilde{C} = \tilde{c} \cdot J_4$ such that $\text{Cay}(\tilde{C}) = R = R_{\tilde{c}}$.

Let us note that the direct calculation gives

$$A.C = -\frac{a \cdot c}{4} T + \frac{a \times c}{2} \cdot J_4. \quad (42)$$

This leads to very similar calculations as in the case of $\text{SU}(2)$ vector parameter. They lead to the following composition of $\text{SO}(3, \mathbb{R})$ matrices in $\text{SO}(4)$

$$\tilde{c} = \langle a, c \rangle_{\text{Cay}_3 j_4} = \left( 1 - \frac{c^2}{4} \right) a + \left( 1 - \frac{a^2}{4} \right) c + \frac{a}{2} \times \frac{c}{2}.$$ 

$$= \frac{1 - 2 \frac{a}{2} \cdot \frac{c}{2} + \frac{a^2 c^2}{4}}{4}. \quad (43)$$
Further development

- We need to check if the $\text{SO}(3, \mathbb{R})$ half-turns are realized as $\text{SO}(n)$ matrices for $n \geq 4$ via the Cayley map applied for the embedded $\mathfrak{so}(3)$ algebra into $\mathfrak{so}(n)$.
- We will investigate what is the composition law for $n > 4$.
- We will investigate if there are efficient formulas to extract the matrix $C$ generating the three-dimensional rotation matrix $R_n(c) \in \text{SO}(n)$.
- We will investigate if some important operators’ representations in dimension $n > 3$ are more convenient.
The Cayley map and higher order representations of rotations


