Star product, star exponential and applications

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Abstract

- We introduce star products for certain function space containing polynomials, and then we obtain an associative, non-commutative or commutative, algebras of functions.
- In this algebra we can consider exponential elements, which are called star exponentials.
- Using star exponentials we can define star functions in the star product algebra.
- This talk is not general, we explain just using concrete examples.
Background

- Weyl, Wigner, Moyal
- BFFLS Formal star product,
- Existence of formal star products
- non-formal star products
The idea of star product is deeply related to the canonical commutation relation in Quantum mechanics, which is given by a pair of operators \( \hat{p}, \hat{q} \) such that

\[
[\hat{p}, \hat{q}] = \hat{p} \hat{q} - \hat{q} \hat{p} = \sqrt{-1} \hbar = i \hbar
\]

where \( \hat{p} = i \hbar \partial_q \) and \( \hat{q} \) is a multiplication operator \( q \times \) acting on the functions of \( q \), and \( \hbar \) is the Planck constant.

The associative non-commutative algebra generated by \( \hat{p} \) and \( \hat{q} \) is called the Weyl algebra which plays a fundamental role in quantum mechanics.
We have another way to give the same algebra without using operators.

The idea is to introduce an associative product into the space of functions of \((q, p)\).

The product is different from the usual multiplication of functions, but is given by a deformation of the usual multiplication in the following way.


The Poisson bracket and biderivation

For smooth functions $f, g$ on $\mathbb{R}^2$ (or $\mathbb{C}^2$), we have the canonical Poisson bracket

$$\{f, g\}(q, p) = \partial_p f \partial_q g - \partial_q f \partial_p g, \quad (q, p) \in \mathbb{R}^2 \text{ (or } \mathbb{C}^2)$$

In deformation quantization, we very often use the notation $\hat{\partial}_p \cdot \hat{\partial}_q - \hat{\partial}_q \cdot \hat{\partial}_p$ such as

$$\{f, g\} = f \left( \hat{\partial}_p \cdot \hat{\partial}_q - \hat{\partial}_q \cdot \hat{\partial}_p \right) g = \partial_p f \partial_q g - \partial_q f \partial_p g$$
The typical star product is the Moyal product given as follows.

For smooth functions $f, g$ we consider a product $f \ast_o g$ given by a power series of the biderivation $\hat{\partial}_p \cdot \hat{\partial}_q - \hat{\partial}_q \cdot \hat{\partial}_p$ such that

$$f \ast_o g = f \exp \frac{i \hbar}{2} \left( \hat{\partial}_p \cdot \hat{\partial}_q - \hat{\partial}_q \cdot \hat{\partial}_p \right) g = f \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i \hbar}{2} \right)^k \left( \hat{\partial}_p \cdot \hat{\partial}_q - \hat{\partial}_q \cdot \hat{\partial}_p \right)^k g$$

$$= fg + \frac{i \hbar}{2} f \left( \hat{\partial}_p \cdot \hat{\partial}_q - \hat{\partial}_q \cdot \hat{\partial}_p \right) g + \frac{1}{2!} \left( \frac{i \hbar}{2} \right)^2 f \left( \hat{\partial}_p \cdot \hat{\partial}_q - \hat{\partial}_q \cdot \hat{\partial}_p \right)^2 g$$

$$+ \cdots + \frac{1}{k!} \left( \frac{i \hbar}{2} \right)^k f \left( \hat{\partial}_p \cdot \hat{\partial}_q - \hat{\partial}_q \cdot \hat{\partial}_p \right)^k g + \cdots$$

The product is well-defined when $f$ or $g$ is a polynomial, and it is easy to see that the product is associative for polynomials.
Now we calculate the commutator of the variables $p$ and $q$. We see

$$p \circ q = p \exp \frac{i\hbar}{2} \left( \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_q - \overleftarrow{\partial}_q \cdot \overrightarrow{\partial}_p \right) q = p \sum_{k=0}^\infty \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k \left( \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_q - \overleftarrow{\partial}_q \cdot \overrightarrow{\partial}_p \right)^k q$$

$$= pq + \frac{i\hbar}{2} p \left( \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_q - \overleftarrow{\partial}_q \cdot \overrightarrow{\partial}_p \right) q = pq + \frac{i\hbar}{2}$$

Similarly we see

$$q \circ p = pq - \frac{i\hbar}{2}$$

Then $p$ and $q$ satisfy the canonical commutation relation under the commutator of the product $\circ$

$$[p, q]_\circ = p \circ q - q \circ p = i\hbar$$
The product $*_o$ is associative on polynomials with canonical commutation relation, and then we obtain the Weyl algebra given by the ordinary polynomials with the product $*_o$.

Using this Weyl algebra of the product $*_o$, we can obtain same results of quantum mechanics and some other extension.

In this talk, we give a brief review on the subject mainly related our investigation: Hideki Omori, Yoshiaki Maeda, Naoya Miyazaki, Akira Yoshioka:


Also see papers in arXiv: math-ph. 1307.0267, etc.
As an application of the star product algebra, we calculate the eigenvalues of the harmonic oscillator by means of the star product $*_{O}$.

**Eigenvalues**
The Schrödinger operator of the harmonic oscillator is

$$\hat{H} = -\frac{\hbar^2}{2} \left( \frac{\partial}{\partial q} \right)^2 + \frac{1}{2} q^2.$$  

The eigenvalues are

$$E_n = \hbar(n + \frac{1}{2}), \quad n = 0, 1, 2, \ldots$$
Star product calculation

We calculate these values $E_n$ by means of the star product $*_{o}$ and functions of $q$ and $p$, parallel to the method in quantum mechanics.

The classical hamiltonian function is

$$H = \frac{1}{2}(p^2 + q^2).$$

We put functions such as

$$a = \frac{1}{\sqrt{2\hbar}}(p + iq), \quad a^{\dagger} = \frac{1}{\sqrt{2\hbar}}(p - iq).$$
Then we calculate the product explicitly and obtain

\[ a^\dagger \star_o a = \frac{1}{2\hbar} (p \star_o p + i[p, q]_\star + q \star_o q) = \frac{1}{2\hbar} (p \cdot p + i \cdot i\hbar + q \cdot q) \]

which shows \( a^\dagger \star_o a = \frac{1}{2\hbar} (p^2 + q^2) - \frac{1}{2} \) and then we have

\[ H = \hbar(N + \frac{1}{2}), \quad (N = a^\dagger \star_o a) \]
The commutator with respect to the star product is easily seen

\[ [a, a^\dagger]_* = a *_o a^\dagger - a^\dagger *_o a = \frac{1}{2\hbar} 2(-i)[p, q]_* = 1 \]

Now we set a function

\[ f_0 = \frac{1}{\pi \hbar} \exp(-2 aa^\dagger) = \frac{1}{\pi \hbar} \exp(-\frac{1}{\hbar}(p^2 + q^2)) \]

and set a function

\[ f_n = \frac{1}{n!} a^\dagger *_o \cdots *_o a^\dagger *_o f_0 *_o a *_o \cdots *_o a \]

for \( n = 0, 1, 2, \cdots \). By a direct calculation we see

\[ a *_o f_0 = f_0 *_o a^\dagger = 0 \]
The relation $[a, a^\dagger]_* = a *_o a^\dagger - a^\dagger *_o a = 1$ induces
$a *_o a^\dagger = a^\dagger *_o a + 1 = N + 1$ and a basic commutation relation

$$N *_o a^\dagger = (a^\dagger *_o a) *_o a^\dagger = a^\dagger *_o (a *_o a^\dagger) = a^\dagger *_o (N + 1)$$

Remark also that $a *_o f_0 = 0$ yields $N *_o f_0 = (a^\dagger *_o a) *_o f_0 = 0$. Then we calculate as

$$N *_o f_1 = N *_o (a^\dagger *_o f_0 *_o a) = a^\dagger *_o (N + 1) *_o f_0 *_o a = f_1$$

By a similar manner we easily see $N *_o f_k = f_k *_o N = k f_k$

Since $H = \hbar (N + \frac{1}{2})$ we have the solutions of the star eigenvalue problem

$$H *_o f_n = f_n *_o H = \hbar (n + \frac{1}{2}) f_n = E_n f_n, \quad (n = 0, 1, 2, \cdots)$$

and thus we obtain the eigenvalues of the harmonic oscillator $\hat{H}$. 
Similarly the star product algebra also gives the exact eigenvalues and their multiplicities for the quantized Kepler problem and more general systems such as the MIC-Kepler problem, the Kepler problem under the influence of the Dirac magnetic monopole.

Generalizing the derivation in the Moyal product, we give general star products as follows (cf. (Omori-Maeda-Miyazaki-Y [9])

§3.1. Examples: Moyal, normal, anti-normal products

The Moyal product is a well-known example of star product.

As in the previous section, we define: for polynomials $f, g$ of the variables $(u_1, \ldots, u_m, v_1, \ldots, v_m)$, the Moyal product $f \ast_o g$ is given by the power series of the biderivation

$$\left( \overleftarrow{\partial_v} \cdot \overleftarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overleftarrow{\partial_v} \right) = \sum_j \left( \overleftarrow{\partial_v j} \cdot \overleftarrow{\partial u_j} - \overleftarrow{\partial u_j} \cdot \overleftarrow{\partial v_j} \right)$$

such that
\[f \ast_o g = f \exp \frac{i\hbar}{2} \left( \vec{\partial}_v \cdot \vec{\partial}_u - \vec{\partial}_u \cdot \vec{\partial}_v \right) g = f \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k \left( \vec{\partial}_v \cdot \vec{\partial}_u - \vec{\partial}_u \cdot \vec{\partial}_v \right)^k g\]

\[= fg + \frac{i\hbar}{2} f \left( \vec{\partial}_v \cdot \vec{\partial}_u - \vec{\partial}_u \cdot \vec{\partial}_v \right) g + \frac{1}{2!} \left( \frac{i\hbar}{2} \right)^2 f \left( \vec{\partial}_v \cdot \vec{\partial}_u - \vec{\partial}_u \cdot \vec{\partial}_v \right)^2 g + \cdots + \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k f \left( \vec{\partial}_v \cdot \vec{\partial}_u - \vec{\partial}_u \cdot \vec{\partial}_v \right)^k g + \cdots\]

Then we have

**Theorem**

*The Moyal product is well-defined on polynomials, and associative.*
Other typical star products are normal product \( \ast_N \), anti-normal product \( \ast_A \) given similarly by

\[
f \ast_N g = f \exp i \hbar \left( \overleftarrow{\partial}_v \cdot \overrightarrow{\partial}_u \right) g, \quad f \ast_A g = f \exp -i \hbar \left( \overleftarrow{\partial}_u \cdot \overrightarrow{\partial}_v \right) g
\]

These are also well-defined on polynomials and associative. By direct calculation we see easily the following.

**Proposition**

(i) For these star products, the generators \((u_1, \ldots, u_m, v_1, \ldots, v_m)\) satisfy the canonical commutation relations

\[
[u_k, v_l]_{\ast_L} = -i \hbar \delta_{kl}, \quad [u_k, u_l]_{\ast_L} = [v_k, v_l]_{\ast_L} = 0, \quad (k, l = 1, 2, \ldots, m)
\]

where \( \ast_L \) stands for \( \ast_O, \ast_N, \ast_A \).

(ii) Then the algebras \((\mathbb{C}[u, v], \ast_L) \quad (L = O, N, A)\) are mutually isomorphic and isomorphic to the Weyl algebra.
Actually the algebra isomorphism

\[ I^N_O : (\mathbb{C}[u,v], *_O) \rightarrow (\mathbb{C}[u,v], *_N) \]

is given explicitly by the power series of the differential operator such as

\[ I^O_N (f) = \exp \left( -\frac{i\hbar}{2} \partial_u \partial_v \right) (f) = \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{i\hbar}{2} \right)^l (\partial_u \partial_v)^l (f) \]

And other isomorphisms are given in the similar form.

**Remark**

*We remark here that these facts correspond to the well-known ordering problem in physics.*
§3.2. General star product

Now by generalizing the biderivations in the previous products, we define a star product on complex domain.

**Biderivation**

Let $\Lambda$ be an arbitrary $n \times n$ complex matrix. We consider a biderivation on $\mathbb{C}^n$

$$\partial_w \Lambda \partial_w = (\partial_{w_1}, \cdots, \partial_{w_n}) \Lambda (\partial_{w_1}, \cdots, \partial_{w_n}) = \sum_{k,l=1}^{n} \Lambda_{kl} \partial_{w_k} \partial_{w_l}$$

where $(w_1, \cdots, w_n)$ is the coordinates of $\mathbb{C}^n$. 
Now we define a star product by the power series of the above biderivation such that

**Definition**

\[ f \star_{\Lambda} g = f \exp \frac{i\hbar}{2} \left( \hat{\partial}_{\Lambda} \hat{\partial}_w \right) g \]

Then similarly as before we see easily

**Theorem**

*For an arbitrary \( \Lambda \), the star product \( \star_{\Lambda} \) is a well-defined associative product on complex polynomials.*
Remark

(i) The star product $\ast_{\Lambda}$ is a generalization of the previous products. Actually

- if we put $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then we have the Moyal product
- if $\Lambda = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, we have the normal product
- if $\Lambda = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$ then the anti-normal product

(ii) If $\Lambda$ is a symmetric matrix, the star product $\ast_{\Lambda}$ is commutative. Furthermore, if $\Lambda$ is a zero matrix, then the star product is nothing but a usual commutative product.
In this section, we fix the antisymmetric part of $\Lambda$ in order to represent the Weyl algebra.

We assume the dimension is even, $n = 2m$. Let $K$ be an arbitrary $2m \times 2m$ complex symmetric matrix. We put a complex matrix

$$\Lambda = J + K$$

where $J$ is a fixed matrix such that

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since $\Lambda$ is determined by the complex symmetric matrix $K$, we denote the star product by $*_K$ instead of $*_\Lambda$. 

\section*{3.3. Star product representation of the Weyl algebra}
We consider polynomials of variables 
\((w_1, \cdots, w_{2m}) = (u_1, \cdots, u_m, v_1, \cdots, v_m)\). By an easy calculation one obtains for an arbitrary \(K\)

**Proposition**

(i) For a star product \(*_K\), the generators \((u_1, \ldots, u_m, v_1, \ldots, v_m)\) satisfy the canonical commutation relations

\[
[u_k, v_l]_* = -i \hbar \delta_{kl}, \quad [u_k, u_l]_* = [v_k, v_l]_* = 0, \quad (k, l = 1, 2, \ldots, m)
\]

(ii) Then the algebra \((\mathbb{C}[u, v], *_K)\) is isomorphic to the Weyl algebra, and the algebra is regarded as a polynomial representation of the Weyl algebra.
Equivalence

As in the case of typical star products, we have algebra isomorphisms as follows.

**Proposition**

For arbitrary star product algebras \((\mathbb{C}[u, v], *_{K_1})\) and \((\mathbb{C}[u, v], *_{K_2})\) we have an algebra isomorphism \(I_{K_1}^{K_2} : (\mathbb{C}[u, v], *_{K_1}) \to (\mathbb{C}[u, v], *_{K_2})\) given by the power series of the differential operator \(\partial_w(K_2 - K_1)\partial_w\) such that

\[
I_{K_1}^{K_2} \left( f \right) = \exp \left( \frac{i \hbar}{4} \partial_w(K_2 - K_1)\partial_w \right) \left( f \right)
\]

where \(\partial_w(K_2 - K_1)\partial_w = \sum_{kl}(K_2 - K_1)_{kl}\partial_{w_k}\partial_{w_l}\).
By a direct calculation we have

**Theorem**

*Then isomorphisms satisfy the following chain rule:*

1. \( I_{K_3}^{K_1} I_{K_2}^{K_3} I_{K_1}^{K_2} = \text{Id} \)
2. \( \left(I_{K_2}^{K_1}\right)^{-1} = I_{K_2}^{K_1} \)

**Remark**

1. **By the previous proposition we see the algebras** \((\mathbb{C}[u, v], \ast_K)\) **are mutually isomorphic and isomorphic to the Weyl algebra. Hence we have a family of star product algebras** \(\{(\mathbb{C}[u, v], \ast_K)\}_K\) **where each element is regarded as a polynomial representation of the Weyl algebra.**

2. **The above equivalences also exist between star products** \(\ast_\Lambda\) **for arbitrary \(\Lambda\)’s with a common skew symmetric part.**
Using polynomial expressions, we can consider exponential elements in the star product algebra.

**Idea of definition.** Here we are considering general star product $*_\Lambda$. For a polynomial $H_*$ of the star product algebra, we want to define a star exponential $e_{i\hbar}^* \frac{t H_*}{i\hbar}$. However, the expansion $\sum_n \frac{t^n}{n!} \left( \frac{H_*}{i\hbar} \right)^n$ of power series of $\frac{H_*}{i\hbar}$ with respect to the star product $*_\Lambda$ is not convergent in general.

Then we define a star exponential by the differential equation.

**Definition**

*The star exponential $e_{i\hbar}^* \frac{t H_*}{i\hbar}$ is given as a solution of the differential equation*

$$\frac{d}{dt} F_t = \frac{H_*}{i\hbar} \ *_\Lambda F_t, \quad F_0 = 1.$$
We are interested in the star exponentials of linear polynomials, and quadratic ones. For simplicity we consider the case $\Lambda = J + K$. For these polynomials, for example we have the following explicit solutions for $*_{\Lambda} = *_{K}$.

**Linear case**

We denote a linear polynomial by $l = \sum_{j=1}^{2m} a_j w_j$. We see

**Proposition**

For $l = \sum_j a_j w_j = \langle a, w \rangle$, the star exponential with respect to the product $*_{\Lambda}$ is

\[
e^{t(l/i\hbar)} = e^{t^2 a K a / 4i\hbar} e^{t(l/i\hbar)}
\]
Quadratic case

Proposition

For a quadratic polynomial $Q_* = \langle wA, w \rangle_*$ where $A$ is a $2m \times 2m$ complex symmetric matrix, we have

$$e_{* A}^{t(Q*/i\hbar)} = \frac{2^m}{\sqrt{\det(I - \kappa + e^{-2i\alpha}(I + \kappa))}} e^{\frac{1}{i\hbar} \langle w \frac{1}{I - \kappa + e^{-2i\alpha}(I + \kappa)}(I - e^{-2i\alpha})J, w \rangle}$$

where $\kappa = KJ$ and $\alpha = AJ$. 
§3.5. Star functions

By the same way as in the ordinary exponential functions, we can obtain several non-commutative or commutative functions using star exponentials.

There are many application of star exponential functions. Today we show examples using a linear star exponentials. (More details, see for example; Omori Maeda Miyazaki Yoshioka Deformation of expression of elements of algebras, MSRI publication 62 (2014). Also see ArXiv OMMY math-ph. e.g., 1307.0267)

In what follows, we consider the star product for the simplest case where $n \times n$ matrix is of the form

$$\Lambda = \begin{pmatrix} \rho & 0 \\ 0 & 0_{n-1} \end{pmatrix}$$

Then we see easily that the star product is commutative and explicitly given by $f \ast_{\Lambda} g = f \exp \left( \frac{i\hbar}{2} \partial_{w_1} \overleftarrow{\partial_{w_1}} \right) g$. 
This means that the algebra is essentially reduced to space of functions of one variable $w_1$.
Thus, we consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and we consider a commutative star product $*_\tau$ with complex parameter $\tau$ such that

$$f(w) *_{\tau} g(w) = f(w)e^{\frac{\tau}{2} \hat{\partial}_w \hat{\partial}_w} g(w)$$

A direct calculation gives that the star exponential of $i \, tw$ with respect to $*_\tau$ is

**Proposition**

$$\exp_{*_\tau} i \, tw = \exp(i \, tw - (\tau/4)t^2)$$
§3.5.1. Star Hermite function

Recall the identity

\[ \exp \left( \sqrt{2}tw - \frac{1}{2}t^2 \right) = \sum_{n=0}^{\infty} H_n(w) \frac{t^n}{n!} \]

where \( H_n(w) \) is an Hermite polynomial. By the explicit formula \( \exp_{\tau} i\, tw = \exp(i\, tw - (\tau/4)t^2) \), we see

\[ \exp_{\tau} (\sqrt{2}tw_{\tau})_{\tau=-1} = \exp \left( \sqrt{2}tw - \frac{1}{2}t^2 \right) \]

Since \( \exp_{\tau} (\sqrt{2}tw_{\tau}) = \sum_{n=0}^{\infty} (\sqrt{2}w_{\tau})^n \frac{t^n}{n!} \) we have

\[ H_n(w) = (\sqrt{2}w_{\tau})^n_{\tau=-1} \]
**Star Hermite function** We define *-Hermite function by

\[ H_n(w, \tau) = (\sqrt{2}w_\tau)^n, \quad (n = 0, 1, 2, \cdots) \]

with respect to \(*_\tau\) product. Then we have

\[ \exp_*(\sqrt{2}tw_\tau) = \sum_{n=0}^{\infty} H_n(w, \tau) \frac{t^n}{n!} \]

**Identities**

Trivial identity \( \frac{d}{dt} \exp_*(\sqrt{2}tw_\tau) = \sqrt{2}w \ast \exp_*(\sqrt{2}tw_\tau) \) for the product \(*_\tau\) yields the identity

\[ \frac{\tau}{\sqrt{2}} H'_n(w, \tau) + \sqrt{2}wH_n(w, \tau) = H_{n+1}(w, \tau), \quad (n = 0, 1, 2, \cdots). \]

for every \( \tau \in \mathbb{C} \).
The exponential law

\[ \exp_\tau(\sqrt{2sw_\ast}) \ast \exp_\tau(\sqrt{2tw_\ast}) = \exp_\tau(\sqrt{2(s + t)w_\ast}) \]

for the product \( \ast_\tau \) yields the identity

\[ \sum_{k+l=n} \frac{n!}{k!l!} H_k(w, \tau) \ast_\tau H_l(w, \tau) = H_n(w, \tau). \]

for every \( \tau \in \mathbb{C} \).
§3.5.2. Star theta function

We can express the Jacobi’s theta functions by using star exponentials.

Recall the formula

\[
\exp_{*\tau} i tw = \exp(i tw - (\tau/4)t^2)
\]

Hence for \(\text{Re } \tau > 0\), the star exponential
\[
\exp_{*\tau} ni w = \exp(ni w - (\tau/4)n^2)
\]
is rapidly decreasing with respect to integer \(n\) and then the summation converges to give

\[
\sum_{n=-\infty}^{\infty} \exp_{*\tau} 2ni w = \sum_{n=-\infty}^{\infty} \exp(2ni w - \tau n^2) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2ni w}, \quad (q = e^{-\tau})
\]
This is convergent and gives Jacobi’s theta function $\theta_3(w, \tau)$. Similarly we have expressions of theta functions as

\[
\begin{align*}
\theta_{1*\tau}(w) &= \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n \exp_{*\tau} (2n+1)i w, \\
\theta_{2*\tau}(w) &= \sum_{n=-\infty}^{\infty} \exp_{*\tau} (2n+1)i w \\
\theta_{3*\tau}(w) &= \sum_{n=-\infty}^{\infty} \exp_{*\tau} 2ni w, \\
\theta_{4*\tau}(w) &= \sum_{n=-\infty}^{\infty} (-1)^n \exp_{*\tau} 2ni w
\end{align*}
\]

Remark that $\theta_{k*\tau}(w)$ is the Jacobi’s theta function $\theta_k(w, \tau)$, $k = 1, 2, 3, 4$ respectively.
We have trivial identities because of the exponential law

\[
\exp_{*\tau} 2i w \ast_{\tau} \theta_{k*\tau} (w) = \theta_{k*\tau} (w) \quad (k = 2, 3)
\]

\[
\exp_{*\tau} 2i w \ast_{\tau} \theta_{k*\tau} (w) = -\theta_{k*\tau} (w) \quad (k = 1, 4)
\]

Then using \( \exp_{*\tau} 2i w = e^{-\tau} e^{2i w} \) and the product formula directly we see the above identities are just

\[
e^{2i w-\tau} \theta_{k*\tau} (w + i \tau) = \theta_{k*\tau} (w) \quad (k = 2, 3)
\]

\[
e^{2i w-\tau} \theta_{k*\tau} (w + i \tau) = -\theta_{k*\tau} (w) \quad (k = 1, 4)
\]
§3.5.3. $\ast$-delta functions

Since the $\ast_{\tau}$-exponential $\exp_{\ast_{\tau}}(itw_{\ast}) = \exp(itw - \frac{\tau}{4} t^2)$ is raidly decreasing with respect to $t$ when $\text{Re} \tau > 0$. Then the integral

$$\int_{-\infty}^{\infty} \exp_{\ast_{\tau}}(it(w - a)_{\ast}) \, dt = \int_{-\infty}^{\infty} \exp(it(w - a) - \frac{\tau}{4} t^2) \, dt$$

converges for any $a \in \mathbb{C}$. We put a star $\delta$-function

$$\delta_{\ast}(w - a) = \int_{-\infty}^{\infty} \exp_{\ast_{\tau}}(it(w - a)_{\ast}) \, dt$$

which has a meaning at $\tau$ with $\text{Re} \tau > 0$. It is easy to see for any element $p_{\ast}(w) \in (\mathbb{C}[w], \ast_{\tau})$,

$$p_{\ast}(w) \ast_{\tau} \delta_{\ast}(w - a) = p(a) \delta_{\ast_{\tau}}(w - a), \quad w_{\ast} \ast_{\tau} \delta_{\ast}(w) = 0.$$
Using the Fourier transform we have

**Proposition**

\[
\begin{align*}
\theta_1^*(w) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \delta^*(w + \frac{\pi}{2} + n\pi) \\
\theta_2^*(w) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \delta^*(w + n\pi) \\
\theta_3^*(w) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta^*(w + n\pi) \\
\theta_4^*(w) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta^*(w + \frac{\pi}{2} + n\pi).
\end{align*}
\]
Now, we consider the $\tau$ with the condition $\text{Re } \tau > 0$. Then we calculate the integral and obtain $\delta_*(w - a) = \frac{2 \sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w - a)^2\right)$.

Then we have

$$
\theta_3(w, \tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta_*(w + n\pi) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w + n\pi)^2\right)
$$

$$
= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \sum_{n=-\infty}^{\infty} \exp\left(-2n\frac{1}{\tau}w - \frac{1}{\tau}n^2\tau^2\right)
$$

$$
= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \theta_3^*(\frac{2\pi w}{i\tau}, \frac{\pi^2}{\tau}).
$$

We also have similar identities for other $\ast$-theta functions by the similar way.
- linear case: star special functions, star Eisenstein series.
- quadratic case: group like object, singularities. etc.
Background

Iwai-Uwano showed that the classical system of MIC-Kepler problem is obtained by the geometric method of $S^1$-reduction, or Marsden-Weinstein reduction for symplectic manifolds.

Star product method uses only classical system with deformed product.
Then by the star product calculation, we expect to deal with the quantized system of MIC-Kepler problem by means of the geometric method of Marsden-Weinstein reduction in natural way.

We discuss this in this subsection.
Now we introduce the MIC-Kepler problem.

We consider a closed two form on $\mathbb{R}^3 = \mathbb{R}^3 - \{0\}$ such that

$$\Omega = \left( q_1 dq_2 \land dq_3 + q_2 dq_3 \land dq_1 + q_3 dq_1 \land dq_2 \right)/r^3$$

where $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$. We consider the cotangent bundle $T^*\mathbb{R}^3$ and a symplectic form

$$\sigma_\mu = dp_1 \land dq_1 + dp_2 \land dq_2 + dp_3 \land dq_3 + \Omega_\mu$$

where $(q, p) = (q_1, q_2, q_3, p_1, p_2, p_3) \in T^*\mathbb{R}^3$ and the 2-form $\Omega_\mu \equiv \mu \Omega$ stands for Dirac’s monopole field of strength $\mu \in \mathbb{R}$. 
Then the MIC-Kepler problem is given as the triple

\[( T^*\mathbb{R}^3, \sigma_\mu, H_\mu ) \]

where \( H_\mu \) is the Hamiltonian function such that

\[
H_\mu (q, p) = \frac{1}{2} \left( p_1^2 + p_2^2 + p_3^2 \right) + \frac{\mu^2}{2r^2} - \frac{k}{r}
\]

and \( k \) is a positive constant.

When \( \mu = 0 \) the system is just the Kepler problem.
The MIC-Kepler problem is obtained by the $S^1$-reduction from the conformal Kepler problem on $T^*\mathbb{R}^4$ (Iwai-Uwano [2]) as follows.

We denote the points by $y \in \mathbb{R}^4$ and $(y, \eta) \in T^*\mathbb{R}^4$.

We identify the point of $T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4)$ by

$$T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4) \mapsto (z_1, z_2, \bar{\zeta}_1, \bar{\zeta}_2) \in T^*\mathbb{C}^2 = \mathbb{C}^4$$

where

$$z_1 = y_1 + iy_2, \quad z_2 = y_3 + iy_4, \quad \bar{\zeta}_1 = \eta_1 + i\eta_2, \quad \bar{\zeta}_2 = \eta_3 + i\eta_4$$
The canonical one form $\theta$ on $T^*\mathbb{R}^4$ is written as

$$\theta(z, \zeta) = \text{Re} \left( \bar{\zeta} \cdot dz \right).$$

The $S^1$ action on the cotangent bundle $T^*\mathbb{R}^4$ is given by

$$\varphi_t : (z, \zeta) \mapsto (e^{it}z, e^{it}\zeta), \quad (t \in \mathbb{R})$$

which preserves the canonical one form $\theta$ and then is an exact symplectic action.

The induced vector field $\nu(z, \zeta)$ on $T^*\mathbb{R}^4$ of the action is

$$\nu(z, \zeta) = (iz, i\zeta)$$

and then a moment map $\psi$ of the action is given by

$$\psi(z, \zeta) = \iota_\nu \theta(z, \zeta) = \text{Im} \left( \zeta \cdot \bar{z} \right) = (\zeta \cdot \bar{z} - \bar{\zeta} \cdot z)/2i$$
$S^1$-reduction

Following the Marsden-Weinstein reduction theory, we consider a level set of the moment map $\psi^{-1}(\mu)$ for $\mu \in \mathbb{R}$. Then the $S^1$-bundle $\pi_\mu : \psi^{-1}(\mu) \to \psi^{-1}(\mu)/S^1$ has the symplectic structure $\omega_\mu$ such that $\iota_\mu^*d\theta = \pi_\mu^*\omega_\mu$, hence we have a reduced symplectic manifold $(\psi^{-1}(\mu)/S^1, \omega_\mu)$, where $\iota_\mu : \psi^{-1}(\mu) \to T^*\mathbb{R}^4$ is the inclusion map. Then one can show

**Proposition (Iwai-Uwano [2])**

The reduced phase space is diffeomorphic to the symplectic manifold of the MIC-Kepler problem,

$$(\psi^{-1}(\mu)/S^1, \omega_\mu) \simeq (T^*\mathbb{R}^3, \sigma_\mu)$$
Now we consider a harmonic oscillator on $T^*\mathbb{R}^4$

$$H(z, \zeta) = \frac{1}{2} |\zeta|^2 + \frac{1}{2} \omega^2 |z|^2$$

Iwai-Uwano [2] introduces the conformal Kepler problem with the Hamiltonian

$$H_{CF}(z, \zeta) = \frac{1}{4|z|^2} (H(z, \zeta) - 4k) - \frac{1}{8} \omega^2 = \frac{1}{8|z|^2} |\zeta|^2 - \frac{k}{|z|^2}$$

The MIC-Kepler problem is the reduced hamiltonian system of the conformal Kepler problem, i.e.,

$$\pi^*_\mu H_\mu = i^*_\mu H_{CF}$$
The conformal Kepler problem is related to the harmonic oscillator on $T^*\mathbb{R}^4$ as

$$4|z|^2\left(H_{CF}(z, \zeta) + \frac{1}{8} \omega^2\right) = H(z, \zeta) - 4k$$

Hence the energy surfaces in $T^*\mathbb{R}^4$ coincide, i.e.,

$$H_{CF} = -\frac{1}{8} \omega^2 \iff H = 4k.$$
Star product calculation of the eigenvalues

On 8-dimensional phase space $T^*\mathbb{R}^4$, we have the canonical Poisson bracket and then by the same way as the previous section, we have the star product $\ast^o$.

We consider functions

\[
\begin{align*}
b_1(z, \zeta) &= \frac{1}{2} \left( \sqrt{\frac{\omega}{\hbar}} z_1 + \frac{i}{\sqrt{\omega \hbar}} \zeta_1 \right), & b_1(z, \zeta)^\dagger &= \overline{b_1(z, \zeta)}, \\
b_2(z, \zeta) &= \frac{1}{2} \left( \sqrt{\frac{\omega}{\hbar}} z_2 + \frac{i}{\sqrt{\omega \hbar}} \zeta_2 \right), & b_2(z, \zeta)^\dagger &= \overline{b_2(z, \zeta)}, \\
b_3(z, \zeta) &= \frac{1}{2} \left( \sqrt{\frac{\omega}{\hbar}} \bar{z}_1 + \frac{i}{\sqrt{\omega \hbar}} \bar{\zeta}_1 \right), & b_3(z, \zeta)^\dagger &= \overline{b_3(z, \zeta)}, \\
b_4(z, \zeta) &= \frac{1}{2} \left( \sqrt{\frac{\omega}{\hbar}} \bar{z}_2 + \frac{i}{\sqrt{\omega \hbar}} \bar{\zeta}_2 \right), & b_4(z, \zeta)^\dagger &= \overline{b_4(z, \zeta)}.
\end{align*}
\]
We see the commutators of these functions are

\[ [b_j, b_k]_\star = [b_j^\dagger, b_k^\dagger]_\star = 0, \quad [b_j, b_k^\dagger]_\star = \delta_{jk} \quad (j, k = 1, 2, 3, 4). \]

We set

\[ N = b_1^\dagger \ast_0 b_1 + b_2^\dagger \ast_0 b_2 + b_3^\dagger \ast_0 b_3 + b_4^\dagger \ast_0 b_4. \]

Then we see

\[ H = \hbar \omega (N + 2), \]

and the moment map \( \psi(z, \zeta) \) is written in terms of \( b_j, b_j^\dagger \) as

\[ \psi(z, \zeta) = \frac{\hbar}{2} (-b_1^\dagger \ast_0 b_1 - b_2^\dagger \ast_0 b_2 + b_3^\dagger \ast_0 b_3 + b_4^\dagger \ast_0 b_4). \]
We put for $j = 1, 2, 3, 4$

$$f_{j,0}(z, \zeta) = \frac{1}{\pi \hbar} e^{-2b_j^\dagger b_j}, \quad f_{j,k}(z, \zeta) = \frac{1}{k!} (b_j^\dagger)^k_{\circ} f_{j,0} \circ (b_j)^k_{\circ}.$$

We consider

$$f_{\vec{n}} = f_{1,n_1} \circ f_{2,n_2} \circ f_{3,n_3} \circ f_{4,n_4}, \quad \vec{n} = (n_1, n_2, n_3, n_4).$$

Parallel to Iwai-Uwano [3], we can calculate the eigenvalues of the MIC-Kepler problem as follows. Similarly as before we easily see

$$H \circ f_{\vec{n}} = \hbar \omega (N + 2) \circ f_{\vec{n}} = \hbar \omega (n_1 + n_2 + n_3 + n_4 + 2) f_{\vec{n}}$$

and

$$\psi \circ f_{\vec{n}} = \frac{\hbar}{2} (-n_1 - n_2 + n_3 + n_4) f_{\vec{n}}$$
Hence the energy level

\[ H_{CF} = -\frac{1}{8} \omega^2 \iff H = 4k \quad \text{and} \quad \psi = \mu \]

is quantized as

\[ 4k = \hbar \omega (n_1 + n_2 + n_3 + n_4 + 2) \]

and

\[ \mu = \frac{\hbar}{2} (-n_1 - n_2 + n_3 + n_4) \]

Thus the quantized energy level of \( H_{CF} \) is

\[-\frac{1}{8} \omega^2 = -\frac{2k^2}{\hbar^2(n_1+n_2+n_3+n_4+2)^2}\]

and the strength of magnetic monopole is quantized as \( \mu = \frac{\hbar}{2} (-n_1 - n_2 + n_3 + n_4) \).
Thus we have

**Theorem**

*The eigenvalues of the MIC-Kepler problem with the strength of magnetic monopole $\frac{m}{2}$ is*

$$E_n = -\frac{2k^2}{\hbar^2(n + 2)^2}, \quad (n \geq |m|, \quad \text{and} \quad n \pm m \equiv 0 \mod 2).$$

*The multiplicity of the eigenvalue $E_n$ is*

$$\frac{(n + m + 2)(n - m + 2)}{4}$$

This is the same as the ones in Iwai-Uwano [3].
References


