Planar $p$-Elasticae and Rotational Linear Weingarten Surfaces

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Introduction

Elastic Curve

Following the model of D. Bernoulli, a curve $\gamma: I \rightarrow \mathbb{R}^2$ is called elastica if it is a critical point of the bending energy $\Theta(\gamma) = \int \kappa^2$. 

Classical Variational Problem. In 1691, J. Bernoulli proposed to determine the final shape of a flexible rod. In 1744, L. Euler published his classification of the planar elastic curves.

Since then, elastica related problems have shown remarkable applications to many different fields: Helfreich-Canham Models in Biophysics, Worldsheets for Kleinert-Polyakov Action in String Theory, Fluid Dynamics..
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4. Remarkable Particular Cases
Planar p-Elasticae

1. Variational Problem
2. Involved Classical Energies
3. Euler-Lagrange Equation
4. Killing Fields along p-Elasticae
5. First Integral of Euler-Lagrange
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Variational Problem

We are going to consider the curvature energy functional

\[ \Theta(\gamma) = \int_{\gamma} (\kappa - \mu)^p \, ds, \]

where \( \mu \) and \( p \in \mathbb{R} \) are fixed real constants, acting on \( \Omega_{p_0} \).

\( \Omega_{p_0} \) is the space of smooth immersed curves of \( \mathbb{R}^2 \) joining two points of it, and verifying that \( \kappa - \mu > 0 \).

Take into account that \( \kappa = \mu \) would be a global minimum if we were considering \( L_1([0, L]) \) as the space of curves.
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Notice that the $p$-Elastic functional

$$\Theta(\gamma) = \int_\gamma (\kappa - \mu)^p ,$$

involves the following classical energies:

- If $p = 0$, we have the Length functional. Critical curves are geodesics.
- If $p = 1$, $\Theta$ is, basically, the Total Curvature functional. Any planar curve is critical.
- If $p = 2$ and $\mu = 0$, $\Theta$ is the Bending energy. And, the critical curves are elastic curves.
- If $p = \frac{1}{2}$ and $\mu = 0$, we have a variational problem studied by Blaschke in 1930, obtaining catenaries.
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Euler-Lagrange Equation

The Euler-Lagrange equation for the curvature energy functional $\Theta(\gamma) = \int_\gamma (\kappa - \mu)^p$, in $\mathbb{R}^2$ with $p \neq 0$, can be written as
\[
d_s^2 \left( (\kappa - \mu)^p - 1 \right) + \kappa^2 (\kappa - \mu)^p - 1 = 0.
\]

Under suitable boundary conditions, solutions of these equations are critical curves for our energy functional. (p-Elastic Curves)

Generalized EMP Equation [3]
The Euler-Lagrange equation is a generalized EMP equation. Indeed, for $p = \frac{1}{2}$, we get the proper EMP equation
\[
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Killing Fields along $p$-Elasticae

A vector field $W$ along $\gamma$, which infinitesimally preserves unit speed parametrization is said to be a Killing vector field along $\gamma$ if it evolves in the direction of $W$ without changing shape, only position. That is, if the following equations hold:

$$W(v)(\bar{t}, 0) = W(\kappa)(\bar{t}, 0) = 0.$$

Killing Vector Fields along $\gamma$[1]

The vector fields along $\gamma$ defined by

$$I = (\kappa - \mu)p - 1B,$$
$$J = ((p - 1)\kappa + \mu)(\kappa - \mu)p - 1T + pds((\kappa - \mu)p - 1)N$$

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Killing Fields along \( p \)-Elasticae

A vector field \( W \) along \( \gamma \), which infinitesimally preserves unit speed parametrization is said to be a Killing vector field along \( \gamma \) if it evolves in the direction of \( W \) without changing shape, only position. That is, if the following equations hold

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\end{align*}
\]

are Killing vector fields along \( \gamma \), if and only if, \( \gamma \) verifies the Euler-Lagrange equation.
First Integral of Euler-Lagrange
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**Theorem [3]**

The derivative of the function $\langle \mathcal{J}, \mathcal{J} \rangle$ along the critical curves is zero. Thus, we have that

$$p^2 |\mathcal{J}|^2 = d,$$

for any positive constant $d$. 
THEOREM [3]

The derivative of the function $\langle \mathcal{J}, \mathcal{J} \rangle$ along the critical curves is zero. Thus, we have that

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Therefore, we can integrate the Euler-Lagrange equation, obtaining

$$(\kappa')^2 = \frac{(\kappa - \mu)^2}{p^2(p - 1)^2} \left( d (\kappa - \mu)^2(1-p) - ((p - 1)\kappa + \mu)^2 \right).$$
Binormal Evolution of p-Elasticae

1. Associated Killing Vector Field
2. Evolution under Binormal Flow
3. Geometric Properties of this Binormal Evolution Surface
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3. Geometric Properties of this Binormal Evolution Surface
A vector field along a curve is a Killing vector field along the curve, if and only if, it extends to a Killing field on the whole $\mathbb{R}^3$. Moreover, this extension is unique. Thus, any planar p-Elasticae has two associated Killing vector fields, which extend $I$ and $J$.

• Killing vector fields in $\mathbb{R}^3$ are the infinitesimal generators of isometries.
• Any Killing vector field in $\mathbb{R}^3$ can be assumed to be of helical type $\lambda_1 X + \lambda_2 V$. 
**Associated Killing Vector Field**

**Unique Extension**

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$$\lambda_1 X + \lambda_2 V.$$
Evolution under Binormal Flow

1. Consider the Killing vector field along $\gamma$ in the direction of the binormal, that is,

$$I = (\kappa - \mu) p - \frac{1}{p} B.$$

2. Let's denote by $\xi$ the associated Killing vector field on $\mathbb{R}^3$ that extends $I$.

3. Since $\mathbb{R}^3$ is complete, we have the one-parameter group of isometries determined by the flow of $\xi$ is given by

$$\{ \phi_t, t \in \mathbb{R} \}.$$

4. Now, construct the surface $S_\gamma := \{ x(s, t) := \phi_t(\gamma(s)) \}$. 
Evolution under Binormal Flow

Take $\gamma$ any planar $p$-Elasticae contained in any totally geodesic surface of $\mathbb{R}^3$. 
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Geometric Properties of this BES

The surface $S_\gamma$ is a $\xi$-invariant surface,
Geometric Properties of this BES

The surface $S_\gamma$ is a $\xi$-invariant surface, and it verifies:

- $S_\gamma$ is a rotational surface.

Theorem [1]
Let $\gamma$ be a planar curve, then, the BES with initial condition $\gamma$ is either, a flat isoparametric surface, if $\kappa$ is constant; or a rotational surface, if $\kappa$ is not constant.

Theorem [4]
Let $\gamma$ be a planar $p$-Elastica, then, the BES generated by $\gamma$ verifies $\kappa_1 = a \kappa_2 + b$, for $a = p^p - 1$, $b = -p^p - 1$. 
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$$a = \frac{p}{p - 1}, \quad b = \frac{-\mu}{p - 1}.$$
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Rotational Linear Weingarten Surfaces

1. Weingarten Surfaces
2. Classification of Rotational Linear Weingarten Surfaces
3. Characterization of Profile Curves
A Weingarten surface in $\mathbb{R}^3$ is a surface where the two principal curvatures $\kappa_1$ and $\kappa_2$ satisfy a certain relation $\Phi(\kappa_1, \kappa_2) = 0$. Well-known families of linear Weingarten surfaces are:

- Umbilical Surfaces (Plane and Sphere)
- Isoparametric Surfaces (Circular Cylinders)
- Constant Mean Curvature Surfaces (Rotational Case: Delaunay Surfaces)
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where $a, b \in \mathbb{R}$, $a \neq 0$. 

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**Classification** \((b = 0)\)

Theorem [4]

The rotational linear Weingarten surfaces satisfying the relation \(\kappa_1 = a \kappa_2\), \(a \neq 0\), are planes, ovaloids and catenoid-type surfaces. Moreover,

- Case \(a > 0\). The rotational surface is an ovaloid.
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(A) \( a < -1 \)

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Let \( a > 0 \) and \( b \neq 0 \). The rotational linear Weingarten surfaces are either ovaloids, circular cylinders or

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- Nodoid-Type Surfaces
A rotational surface $M$ can be, locally, described by

$$M = S_\gamma := \{ x(s, t) = \phi_t(\gamma(s)) \},$$

where,

- $\phi_t$ is the rotation,
- $\gamma(s)$ is the profile curve (that is, the curve everywhere orthogonal to the orbits of $\phi_t$).

Then,

**Theorem [4]**

Let $M$ be a rotational linear Weingarten surface and let $\gamma(s)$ be its profile curve. Then, if $a \neq 1$, $\gamma$ is a planar $\mu$-Elastic curve for

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Summary of the Main Results

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\text{BES + Planar p-Elastica} \iff \text{Rotational LW}
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Binormal evolution surfaces generated from planar p-Elasticae, are precisely, rotational linear Weingarten surfaces with \( a \neq 1 \).

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Every rotational linear Weingarten surface (with \( a \neq 1 \)) admits a geodesic foliation by planar p-Elasticae.

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Remarkable Particular Cases

1. Classic Elastic Curves and Mylar Balloons
2. Extended Blaschke’s Energy and Delaunay Surfaces
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Take $p = 2$ and $\mu = 0$ in the $p$-elastic energy. That is, we have the bending energy

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**Curvature of Planar Elastic Curves**

Solving the Euler-Lagrange equations, we obtain that the non-geodesic planar elastic curves have curvature given by

$$\kappa(s) = \kappa_o \operatorname{cn} \left( \frac{\kappa_o}{\sqrt{2}} s, \frac{\sqrt{2}}{2} \right).$$
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- $\kappa_o = \kappa_o(d)$ is a constant (the maximum curvature) and $cn$ denotes the Jacobi cosine.
**Profile Curves of Mylar Balloons**

The binormal evolution surface generated from a planar elastic curve is a rotational surface verifying $\kappa_1 = 2\kappa_2$. 
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- These rotational surfaces are essentially unique (up to translations and homotheties). They are called Mylar Balloons.
- We also know that, planar elastic curves verify \( x(s) = \frac{2\kappa(s)}{\sqrt{d}} \).
- Thus, after rotating we obtain the parametrization of Mylar Balloons:

\[
x(s, \theta) = \frac{1}{\sqrt{d}} \left( 2\kappa \cos \theta, 2\kappa \sin \theta, \int \kappa^2 \, ds \right),
\]

where \( \kappa(s) \) is the curvature of \( \gamma \).
Take $p = \frac{1}{2}$ in the $p$-Elastic energy, that is,

$$\Theta(\gamma) := \int_{\gamma} \sqrt{\kappa - \mu} = \int_{0}^{L} \sqrt{\kappa(s) - \mu} \, ds.$$
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1. If $\kappa = \mu$ then $\gamma$ is an absolute minima for $\Theta$.
2. Now, let $\gamma$ be a non-constant curvature critical curve. Then,

$$\kappa(s) = \frac{4d}{1 + 16d^2s^2},$$

for every $d > 0$ if $\mu = 0$. 
Extended Blaschke’s Energy

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for every $d > 0$ if $\mu = 0$. Or, if $\mu \neq 0$,

$$
\kappa(s) = \frac{2\mu(\omega^2 + \omega \sin 2\mu s)}{1 + \omega^2 + 2\omega \sin 2\mu s} ,
$$

where $\omega^2 = 1 + \frac{\mu}{d}$.
Case $\kappa = \mu$. 

Critical curves are either lines ($\mu = 0$) or circles. They are roulettes of conic foci.

For the critical curves with non-constant curvature we have Geometric Characterization [2]

Non-constant curvature critical curves for the extended Blaschke’s energy in $\mathbb{R}^2$ are, precisely, the roulettes of conic foci with non-constant curvature.

- If $\mu = 0$, we have catenaries.
- If $\mu \neq 0$ and $\omega < 1$, they are nodaries.
- If $\mu \neq 0$ and $\omega > 1$, they are undularies.
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**Characterization of Delaunay Surfaces** [2]

A Delaunay surface is, precisely, a binormal evolution surface with a critical curve for the extended Blaschke’s energy as initial condition.
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**Characterization of Delaunay Surfaces** [2]

A Delaunay surface is, precisely, a binormal evolution surface with a critical curve for the extended Blaschke’s energy as initial condition. Moreover, the constant mean curvature is given by

$$H = -\mu.$$


4. R. López and A. Pámpano, Classification of Rotational Surfaces in Euclidean Space Satisfying a Linear Relation Between their Principal Curvatures, submitted.
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