JGSP 18 (2010) 63–86

JOURNAL OF
Geometry and Symmetry
in Physics

FROM GENERALIZED KÄHLER TO GENERALIZED SASAKIAN STRUCTURES

IZU VAISMAN

Communicated by Ivaïlo M. Mladenov

Abstract. This is an introductory paper that provides a first introduction to geometric structures on $TM \oplus T^*M$. It contains definitions and characteristic properties (some of them new) of generalized complex, Kähler, almost contact (normal, contact) and Sasakian manifolds.

1. Introduction

This is an expository paper. Its aim is to introduce the reader into the new subject of generalized structures. The non-previously published results are Proposition 17 and Theorem 24, which give new characterizations of generalized, normal, almost contact and generalized, Sasakian structures, and some remarks about non degenerate, generalized, almost contact structures.

The word "generalized" has the following precise meaning. If M is m-dimensional, differentiable manifold, a "classical structure" on M is a reduction of the structure group of the tangent bundle TM from the general linear group $\mathrm{GL}(m, \mathbb{R})$ to a certain subgroup G. The "generalization" consists in replacing TM by the *big tangent bundle* $T^{\mathrm{big}}M = TM \oplus T^*(M)$. The bundle $T^{\mathrm{big}}M$ has a natural, neutral metric (non degenerate, signature zero) g defined by

$$g((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) + \beta(X)), \qquad X \in \chi(M), \quad \alpha \in \Omega^1(M). \quad (1)$$

Hence, the natural structure group of $T^{\mathrm{big}}M$ is the group $\mathrm{O}(m, m)$ that preserves the canonical neutral metric on $\mathbb{R}^{2m}$ and the "generalized structures" will be reductions of the structure group $\mathrm{O}(m, m)$.

Furthermore, classical integrability conditions are expressed in terms of the Lie bracket of vector fields on M. A generalized bracket is the *Courant bracket* [6] given by the formula

$$[(X, \alpha), (Y, \beta)] = ([X, Y], L_X\beta - L_Y\alpha + \frac{1}{2}\mathrm{d}(\alpha(Y) - \beta(X))). \quad (2)$$