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SYMMETRY AND SOLUTIONS TO THE HELMHOLTZ EQUATION INSIDE AN EQUILATERAL TRIANGLE

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Abstract. Solutions to the Helmholtz equation within an equilateral triangle which solve either the Dirichlet or Neumann problem are investigated. This is done by introducing a pair of differential operators, derived from symmetry considerations, which demonstrate interesting relationships among these solutions. One of these operators preserves the boundary condition while generating an orthogonal solution and the other leads to a bijection between solutions of the Dirichlet and Neumann problems.

MSC: 35J05, 35B06
Keywords: Dirichlet and Neumann problems, equilateral triangle, Helmholtz equation, Laplacian

1. Introduction

The solutions to many important physical problems, such as electromagnetic waves in waveguides [13], lasing modes in nanostructures [5], the electronic structure of graphene [10] and the quantum eigenvalues and eigenfunctions for various potential energies [7] are obtained by solving the ubiquitous Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0. \tag{1}$$

Studying the solutions to this equation is both a very old problem and one which continues to be an area of active research ([2], [14], [16]). In this paper, we discuss the solutions to this equation when the region of interest is an equilateral triangle and we consider two different boundary conditions: Dirichlet and Neumann. Although the explicit solutions in these cases are well-known, ([5], [7], [8], [12], [3], [4]) we present an alternative and more elegant framework for understanding them. These insights follow from properties of two differential operators which exploit the symmetry of the boundary.

The first operator we introduce, Θ_s, will preserve the boundary condition but transforms the solution to an orthogonal one. For degenerate solutions this becomes a