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CASSINI OVALS IN HARMONIC MOTION ORBITS*

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Abstract. We discover the appearance of interesting Cassinian ovals in the motion of a two-dimensional harmonic oscillator. The trajectories of the oscillating points are ellipses depending on a parameter. The locus of the foci of these ellipses is a Cassini oval. The form of this oval depends on the magnitude of the initial velocity.

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1. Introduction

In this note we point out an interesting geometric phenomenon. We consider mechanical vibrations on the plane where the vibrating point traces various ellipses. We show that the foci of these ellipses trace different Cassini ovals. The forms of these ovals depend only on the initial velocity.

The simple free non-damped harmonic oscillator is governed by the differential equation

$$x'' + \omega^2 x = 0 \tag{1}$$

where $x(t)$ is the position function, $t \geq 0$ is time, and ω is the angular frequency. This equation results from Newton's law $F = m\,a$ by using the force $F = -k\,x$, where $k > 0$ is a constant. Then we have $m\,x'' = -k\,x$ or (1) with $\omega^2 = k/m$.

Suppose the initial position is the point $(\mathring{x}, 0)$ in the xy-plane and the initial velocity is $\mathring{v} = x'(0)$. The law of motion is

$$x(t) = \mathring{x}\cos(\omega t) + \frac{\mathring{v}}{\omega}\sin(\omega t). \tag{2}$$

The point $M(x(t), 0)$ oscillates over the interval $[-c, c]$, where $c = \sqrt{\mathring{x}^2 + \dfrac{\mathring{v}^2}{\omega^2}}$.

Now we extend this motion to two dimensions by assuming that the initial velocity is a vector $\mathring{\mathbf{v}} = (\mathring{v}\cos\alpha,\ \mathring{v}\sin\alpha)$, cutting angle α with the x-axis. The motion

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*Dedicated to the memory of Professor Vasil V. Tsanov 1948-2017.

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