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ON THE GEOMETRY OF AXISYMMETRIC VESICLES

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Abstract. Vesicle shapes with axial symmetry are modeled using Cassinian oval which when rotated leads to variety of surfaces specified as the level sets of algebraic function. A continuous set of shapes including sphere, torus, biconcave discocyte and dumbbell are described by considering Cassinian oval with a focus distance that is a purely imaginary number. Although these surfaces are not exact solutions of the vesicle shape equation, they reveal some qualitative geometric properties that are difficult to examine by currently available numerical or exact solutions. Relation between the volume of the vesicle at constant surface area and homogeneity index is derived. Going from the implicit to the explicit coordinatization of these surfaces, their fundamental forms and curvatures are found in a form convenient for experimentalists. The integrals specifying the volume and surface area are evaluated analytically in a form relevant to the sphere geometry. The free energy corresponding to bending of outer membrane of the vesicles, modeled by the Cassinian oval, is plotted numerically as a function of the form factor or the reduced volume. The effect of the so called Helfrich spontaneous curvature on the energy minimization is also examined.

1. Introduction

Although the description of vesicle shapes have been a subject of a research interest for a long time (cf. Seifert and Lipowsky [18] for quite representative but still unexhaustive review), the available up to now explicit analytical solutions of the vesicle shape equation are only seven (cf. Ou-Yang *et al.* [15]). This can be related mainly to the fact that we are dealing with a rather complicated differential equation that is not well studied in mathematics. Quite recently, a solution in the case of axisymmetric vesicles has been found which can describe a continuous set of shapes, from red blood cell biconcave discocyte to ellipsoid, sphere, and capped cylinder (cf. Liu *et al.* [13]). Looking closely at