

VARIATIONAL PROBLEMS IN ELASTIC THEORY OF BIOMEMBRANES, SMECTIC-A LIQUID CRYSTALS, AND CARBON RELATED STRUCTURES

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Abstract. After a brief introduction to several variational problems in the study of shapes of thin structures, we deal with variational problems on two-dimensional surface in three-dimensional Euclidian space by using exterior differential forms and the moving frame method. The morphological problems of lipid bilayers and stabilities of cell membranes are also discussed. The key point is that the first and the second order variations of the free energy determine equilibrium shapes and mechanical stabilities of structures.

1. Introduction

The morphology of thin structures (always represented by a smooth surface M in this paper) is an old problem. First, we look back to the history [21]. As early as in 1803, Plateau has studied soap films by attaching them to a metallic ring when the ring passes through soap water [25]. By taking the minimum of the **free energy** $F = \lambda \int_M dA$, he obtained the equation $H = 0$, where λ and H are the surface tension and mean curvature of the soap film, respectively. From 1805 and 1806, Young [37] and Laplace [11] studied soap bubbles. By taking the minimum of the free energy $F = p \int dV + \lambda \int_M dA$, they obtained $H = p/2\lambda$, where p is the osmotic pressure (pressure difference between outer and inner sides) of a soap bubble and V is the volume enclosed by the bubble. We can only observe spherical bubbles because “An embedded surface with constant mean curvature in 3-dimensional (3D) Euclidian space (\mathbb{E}^3) must be a spherical surface” [1]. In 1812, Poisson [26] considered a solid shell and put forward the free energy $F = \int_M H^2 dA$. Its Euler-Lagrange equation is $\nabla^2 H + 2(H^2 - K)H = 0$ [36]. Now the solutions to this equation are called **Willmore surfaces**. In 1973, Helfrich recognized that lipid bilayers could be regarded as **smectic-A (SmA) liquid crystals (LCs)** at room