Geometric Structures in Four-dimension and Almost Hermitian Structures

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Abstract. It is known [25], [26] (cf. also, [5]) that among nineteen four-dimensional geometries (in the sense of Thurston [23]), there are fourteen geometries which admit complex structure compatible with a group of isometries. We focus our attention to the five geometries which do not admit such a complex structure, and analyze if these geometries can admit or not an almost Hermitian structure. We discuss the existence problem of an almost Hermitian structure on these five geometries from a general point of view. We consider also a question: can these geometries admit an almost Hermitian structure compatible with the geometric structure? Our analysis will be made from two different points of view, i.e., a Riemannian version and a Neutral version.

Keywords: four-dimensional geometries, group of isometries, almost Hermitian structures, opposite almost Hermitian structure, almost Kähler structure, opposite almost Kähler structure, complex structures, neutral metrics
PACS: 02.40.-k, 02.40.Ma, 02.40.Tt

INTRODUCTION

By *geometry* in the sense of Thurston [23] (see also [21]), we mean a pair (X, G_X) with X a simply-connected manifold, G_X a Lie group acting transitively on X, such that

1. The stabilizer subgroup K_X in G_X of a point in X is constant (equivalently, X has a G_X-invariant Riemannian metric).
2. G_X has discrete subgroups Γ such that $\Gamma\backslash X$ (or equivalently, $\Gamma\backslash G_X$) has finite volume, i.e., Γ is a lattice in the sense of [20].

Thanks to Thurston [23], it is well-known that the three-dimensional geometries are classified into eight classes

$$\begin{gathered} S^3, \quad E^3, \quad H^3 \\ S^2\times E^1, \quad H^2\times E^1, \quad Nil^3, \quad \widetilde{SL}(2,\mathbb{R}), \quad Sol^3. \end{gathered} \tag{1}$$

There are nineteen classes [25], [26] (cf. also [5]) of four-dimensional geometries

$$\begin{gathered} S^4, \quad E^4, \quad H^4, \quad P^2\mathbb{C}, \quad H^2\mathbb{C} \\ S^2\times S^2, \quad S^2\times E^2, \quad S^2\times H^2, \quad E^2\times H^2, \quad H^2\times H^2 \\ S^3\times E^1, \quad H^3\times E^1, \quad \widetilde{SL}_2\times E^1, \quad Nil^3\times E^1 \\ Sol^4_o, \quad F^4, \quad Nil^4, \quad Sol^4_{m,n} \text{ (including } Sol^3\times E^1), \quad Sol^4_1. \end{gathered} \tag{2}$$

International Workshop on Complex Structures, Integrability and Vector Fields
AIP Conf. Proc. 1340, 66-80 (2011); doi: 10.1063/1.3567126