MODULAR STRUCTURES IN GEOMETRIC QUANTIZATION

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Abstract

The purpose of this lecture is to show how certain modular structures, borrowed from the theory of von Neumann algebras, can be exploited to extract primary representations (prequantization) and irreducible representations (quantization) from the regular representation of the symmetry group of the physical systems to be considered. The emphasis is on presenting specific examples for which the solution is exhibited explicitly.

1. REVIEW AND MOTIVATION

Depending on who is speaking, physicist or mathematician, geometric quantization is a method either to select "relevant" representations, or to construct "specific" representations of a group; the matter of which group one wishes to consider is a further question of idiosyncratic (or "classical") preferences.

The general idea of geometric quantization was discovered from a physical approach by Souriau,[1] and was formulated independently by Kirillov.[2] Kostant [3] made important contributions; see also the expository accounts by: Simms & Woodhouse,[4] Guillemin & Sternberg,[5] Abraham & Marsden,[6] Śniatycki,[7] Woodhouse.[8]

In this lecture, we want to draw attention to some, hitherto neglected, algebraic structures underlying the geometric quantization programme; to be specific, we are going to focus on the following examples: the Weyl group of the canonical commutation relations, first for the flat configuration space $\mathbb{R}^n$, and its generalisation to the simply connected manifold $\mathbb{H}^n$ of constant negative curvature; we will then show how much of what one learns from these two examples can be transferred to certain representations of groups such as $SO(2,3)$. For the moment, let us denote by: G any of these groups; $d\mu$ its left-invariant Haar measure; and $U^L(G)$ its left-regular representation on the Hilbert space $L^2(G,d\mu)$.

The Kirillov[2] method of co-adjoint orbits extracts from $U^L(G)$ unitary sub-representations $U_P(G)$; these are best understood as operating on the Hilbert space $L^2(\Omega_{\xi^*}, \omega\wedge\cdots\wedge\omega)$ where Ω_{ξ^*} is a coadjoint orbit of G, i.e. the orbit of some $\xi^*\in\mathcal{G}^*$ under the coadjoint-action of G on the dual $\mathcal{G}^*$ of its Lie algebra $\mathcal{G}$; ω is the symplectic form naturally [2] induced by the coadjoint action of G on Ω_{ξ^*}. The physical interpretation of this procedure is that the symplectic manifold (Ω_{ξ^*},ω) models a classical phase space, e.g. in the simplest cases, the cotangent bundle T^*M of some configuration