LOOP VARIABLES IN QUANTUM GRAVITY AND VASSILIEV INVARIANTS

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Abstract

Some mathematical topics connected, directly or indirectly, with non-local variables in gravity are briefly discussed. These topics are: 1) the correspondence between connections and holonomies, 2) Chen integrals and the group of loops, 3) Vassiliev invariants and Kontsevich integrals.

1. INTRODUCTION

The central mathematical theme underlying the topics presented here is the concept of *holonomy*. Let M be a smooth, connected and paracompact manifold, $P \xrightarrow{\pi} M$ a principal-G bundle with connection, $*$ a point on M, p_0 a point in the fibre over $*$ and $\gamma : [0,1] \to M$ a piecewise smooth loop based at $*$, i.e. $\gamma(0) = \gamma(1) = *$. (Throughout this article all maps referred to as piecewise smooth are understood to be continuous as well.) Then γ has a unique horizontal lift starting at p_0, which will be denoted $\gamma^\uparrow$. The holonomy of the connection around γ relative to p_0, denoted $\mathcal{H}(\gamma)$, is the element of G given by $\gamma^\uparrow(0) = \gamma^\uparrow(1)\mathcal{H}(\gamma)$. (The alternative, more frequently used convention, $\gamma^\uparrow(1) = \gamma^\uparrow(0)\mathcal{H}(\gamma)$, is slightly less convenient for the purposes of this article.)

In a local patch $U \subset M$ the connection may be described by an $L(G)$-valued one-form A, where $L(G)$ stands for the Lie algebra of G. If G is identified with its image under a faithful matrix representation and the image of γ is contained in U, the holonomy may be defined in terms of an initial-value problem for a function $g : [0,1] \to G$, namely

$$\left.\begin{array}{rcl} \dot{g}(t) + A(t)g(t) & = & 0 \\ g(0) & = & I \end{array}\right\} \rightsquigarrow \mathcal{H}(\gamma) = g(1)^{-1} \tag{1.1}$$

where $A(t)dt = \gamma^* A(t)$ and I is the identity (matrix) of G. The initial-value problem is formally solved by the path-ordered exponential:

$$g(t) = \mathcal{P}\exp\int_\gamma^t A := I + \sum_{n=1}^\infty \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, A(t_1)A(t_2)\cdots A(t_n) \tag{1.2}$$

where the path-ordering prescription replaces the natural integration region of the nth term in the expansion of the exponential, i.e. $[0,t]^n$, by the subsimplex $0 \le t_1 \le \ldots \le t_n \le t$, instead of multiplying by $1/n!$.

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